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LE MASSE ED ANGOLI DI MESCOLAMENTO NELLA TEORIA DI GRANDE UNIFICAZIONE BASATA SU $SU(5)$

NEUTRINO MASSES AND MIXINGS IN $SU(5)$ GRAND UNIFIED THEORY

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Abstract

In questa tesi si discute l'origine delle piccole masse dei neutrini attraverso il cosiddetto meccanismo seesaw. Esso porta a piccole masse di Majorana, il che implica violazione del numero leptonico di due unita', come ad esempio nel caso del decadimento beta doppio senza neutrini o nel caso di fisica ai collider con produzione di di-leptoni dello stesso segno accompagnati da jet.

Ci si focalizzera' su una teoria di grande unificazione $SU(5)$ minimale ma realistica, che implica una nuova rappresentazione fermionica aggiunta oltre l'originaria teoria di Georgi-Glashow. Questa teoria porta ad un neutrino senza massa e, soprattutto, ad un tripletto fermionico di $SU(2)$ leggero, chimatato T, con $M_T < 1TeV$. La fenomenologia del tripletto e' ricca sia sotto il profilo di nuova fisica ai collider che per quanto rigurda decadimenti rari.

Si calcoleranno i decadimenti del tripletto e si studieranno in dettaglio tutti i processi con violazione del numero leptonico, come ad esempio $\mu \to e\gamma$ o $\mu \to 3e$.

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Chapter 1

Introduction

Recent observation has detected oscillations of neutrinos coming from the sun and the atmosphere. From these observations, two neutrino mass differences have been measured[1], the values are $\Delta m^2_{21} = (8.0 \pm 0.3) \times 10^{-5}\,\mathrm{eV}^2$ from solar neutrino experiment and $\Delta m^2_{23} = 1.9\,\mathrm{to}\,3.0 \times 10^{-3}\,\mathrm{eV}^2$ from atmospheric neutrino experiment. From this result we conclude that at least two neutrinos must be massive.

To create massive neutrino in the Standard Model we can simply introduce a right-handed neutrino to the model and create the usual Dirac neutrinos like we do with other fermions. However, in general gauge invariance also allows for Majorana mass[2]. As we will see later in this work, when we consider neutrino as Majorana particle, the left-handed and right-handed states may have different mass. Moreover, through the seesaw mechanism we can eventually show that the left-handed mass should be proportional to the inverse of the right-handed mass.

Now if only the right-handed states are heavy, seesaw mechanism should be able to explain the smallness of left-handed neutrino mass. To make a justification for heavy right-handed neutrino we will need to go to the theory beyond the Standard Model. In this work we are going to use the Grand Unified Theory based on $SU(5)$ group.

The main idea behind Grand Unified Theory is to unify the electromagnetic, weak and strong force at some point at high energy called GUT scale. The central of grand unification are proton decay and the existence of magnetic monopoles. The minimal such theory is the $SU(5)$ model of Georgi and Glashow [3]. Unfortunately, in this minimal model the gauge couplings fail to unify and equally important neutrinos are still massless as in the minimal Standard Model.

We are then forced to modify the minimal $SU(5)$ theory. The simplest possibility is to add a fermionic adjoint representation[4, 5] which cures both problems. The unification is achieved through a light $SU(2)_L$ triplet, and its neutral component together with a singlet fermion in the adjoint acts as the right-handed neutrinos which are going to produce a realistic neutrino mass matrix through the seesaw mechanism.

The most interesting feature of this model is the prediction of light $SU(2)_L$ triplet fermion which can be detected in near future experiment through the exciting possibility of lepton number violation. More precisely, the decays of the triplet produce some sign di-leptons accompanied by jets. Also, the light triplet allows for lepton flavour violation (LFV) [6], i.e. processes such as $\mu \to e\gamma, \tau \to \mu\gamma, \mu \to ee\bar{e}$, etc. In other words, this theory leads to a possibility of probing the origin of neutrino mass at colliders such as LHC.

In this work, we review the Majorana and Dirac neutrinos of fermions, the seesaw mechanism for neutrino mass and the salient features of the $SU(5)$ Grand Unified Theory. We also compute a number of LFV processes that can be probed in the near future.

The rest of this thesis is organized as follow,

- In chapter 2, we review the theory of Majorana particle to give us the basics for doing the seesaw mechanism
- In chapter 3, we explain the seesaw mechanism. We also discuss the lepton mixing, both in Dirac and Majorana cases.
- In chapter 4, we review the $SU(5)$ Grand Unified Theory.

- In chapter 5, we cure the problems of the minimal $SU(5)$ through the adjoint fermion and discuss the hybrid seesaw type I and III.

- In chapter 6, we calculate the decay rate of processes such as triplet decay, $e^i \to e^j \gamma$ and $e^i \to 3e^j$ based on the above theory. We discuss the predictions that can be experimentally tested.

- In chapter 7, we summarize our findings and offer concluding remarks.

Chapter 2

Majorana Particles

In the minimal Standard Model, the charged fermions are the 2-component massless Weyl spinors, which obtain their masses throught the Higgs mechanism. In the process they become four component Dirac fermions. We will try to imitate this for neutrinos by adding the right-handed singlet ones. However, in general this will lead to Majorana neutrinos, and the smallness of left-handed neutrino mass will result from the large mass of the right-handed counterpart. In what follows, we first discuss the notions of Dirac and Majorana masses[7].

2.1 Dirac and Majorana mass

Under Lorentz transformation, a Dirac spinor ψ transform as[8]

$$\psi \to \psi' = \exp(-\frac{i}{2}\sigma_{\mu\nu}\omega^{\mu\nu})\psi \tag{2.1}$$

Where $\omega^{\mu\nu}$ is an antisymmetric parameter and $\sigma_{\mu\nu}$ is the Lorentz transformation generator for Dirac spinor which can be defined by Dirac gamma matrices γ_μ

$$\sigma_{\mu\nu} = \frac{1}{4i}[\gamma_\mu, \gamma_\nu] \tag{2.2}$$

σ_{0i} is the generator for Lorentz boost in i direction, while σ_{ij} is the generation for rotation in ij plane.

We will use the Weyl representation for the gamma matrices

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.3}$$

so that we have

$$\sigma_{0i} = \frac{i}{2} \begin{pmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{pmatrix}, \quad \sigma_{ij} = \frac{\epsilon_{ijk}}{2} \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix} \tag{2.4}$$

From (2.4) it is clear that ψ is reducible, we can then express it as a combination of two irreducible spinors. We write

$$\psi = \begin{pmatrix} u_L \\ u_R \end{pmatrix} \tag{2.5}$$

Where u_L and u_R are 2-component complex spinors which transform under Lorentz transformation according to

$$\begin{aligned} u'_L &= \exp\left(\vec{\beta} \cdot \frac{\vec{\sigma}}{2} - i\vec{\theta} \cdot \frac{\vec{\sigma}}{2}\right) u_L \\ u'_R &= \exp\left(-\vec{\beta} \cdot \frac{\vec{\sigma}}{2} - i\vec{\theta} \cdot \frac{\vec{\sigma}}{2}\right) u_R \end{aligned} \tag{2.6}$$

where β and θ correspond to boost and rotation respectively.

From (2.6) we have two ways of making Lorentz invariants (note the identity $\sigma\sigma^2 = -\sigma^2\sigma^*$)

$$\begin{aligned} \text{Dirac mass term} &: m_D u_L^\dagger u_R \\ \text{Majorana mass term} &: m_M u_L^T i\sigma_2 u_L \end{aligned} \tag{2.7}$$

As can be seen that unlike in the case of Dirac mass, the Majorana mass is made from purely one chirality. In other words, in the case of pure Majorana particles we will have no mixing between left and right particle so that we may have a 2 components massive particles. It is important to stress that the invariants in Eq. (2.7) are created based only on Lorentz symmetry. On the other hand, Majorana mass terms clearly violates the conservation of an internal quantum number.

2.2 More on Majorana particles

Consider a pure Majorana particle, the full Lagrangian can be written in 2-component notation as

$$\mathcal{L}_M = iu_L^\dagger(\partial_0 + \bar{\sigma}\cdot\bar{\nabla})u_L - \frac{m_M}{2}(u_L^T i\sigma^2 u_L - u_L^\dagger i\sigma^2 u_L^*) \tag{2.8}$$

The factor $\frac{1}{2}$ in the mass term is put for later convenience. To write it in the usual 4-component notation, we define

$$\psi_M \equiv \begin{pmatrix} u_L \\ -i\sigma^2 u_L^* \end{pmatrix} = \psi_L + \psi_R \tag{2.9}$$

Where

$$\psi_L = \begin{pmatrix} u_L \\ 0 \end{pmatrix}; \psi_R = \begin{pmatrix} 0 \\ -i\sigma^2 u_L^* \end{pmatrix} \tag{2.10}$$

Note that ψ_R does transform as a right handed particle.

Now we will have

$$\begin{aligned}\bar{\psi}_M\psi_M &= \begin{pmatrix} u_L^\dagger & u_L^T i\sigma^2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} u_L \\ -i\sigma^2 u_L^* \end{pmatrix} \\ &= u_L^T i\sigma^2 u_L - u_L^\dagger i\sigma^2 u_L^* \end{aligned} \tag{2.11}$$

and

$$\begin{aligned}\bar{\psi}_M\gamma^\mu\partial_\mu\psi_M &= \begin{pmatrix} u_L^\dagger & u_L^T i\sigma^2 \end{pmatrix}\begin{pmatrix} \partial_0 + \vec{\sigma}\cdot\vec{\nabla} & 0 \\ 0 & \partial_0 - \vec{\sigma}\cdot\vec{\nabla} \end{pmatrix}\begin{pmatrix} u_L \\ -i\sigma^2 u_L^* \end{pmatrix} \\ &= u_L^\dagger(\partial_0 + \vec{\sigma}\cdot\vec{\nabla})u_L + u_L^T\sigma^2(\partial_0 - \vec{\sigma}\cdot\vec{\nabla})\sigma^2 u_L^* \\ &= u_L^\dagger(\partial_0 + \vec{\sigma}\cdot\vec{\nabla})u_L + u_L^T(\partial_0 + \vec{\sigma}^*\cdot\vec{\nabla})u_L^* \\ &= u_L^\dagger(\partial_0 + \vec{\sigma}\cdot\vec{\nabla})u_L + u_L^T(\partial_0 + \vec{\sigma}^T\cdot\vec{\nabla})(u_L^\dagger)^T \\ &= 2u_L^\dagger(\partial_0 + \vec{\sigma}\cdot\vec{\nabla})u_L \end{aligned} \tag{2.12}$$

Therefore Eq. (2.8) can be written as

$$\mathcal{L}_M = \frac{1}{2}\left(i\bar{\psi}_M\gamma^\mu\partial_\mu\psi_M - m_M\bar{\psi}_M\psi_M\right) \tag{2.13}$$

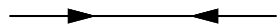


Figure 2.1: propagator for a majorana particle, the double pointing arrow indicates the ambiguity between particle and antiparticle

which is the usual Dirac Lagrangian (the factor $\frac{1}{2}$ in front is insignificant). Therefore the propagator for Majorana particle will have the same form as Dirac propagator

$$\frac{i}{\not{p} - m_M} \tag{2.14}$$

We will now look onto the physical consequence of the definition used in Eq. (2.9). Notice that

$$\psi_M = i\gamma^2\psi_M^* \tag{2.15}$$

In other word, a majorana particle is equivalent to its antiparticle.

The Eq. (2.15) is sometimes called the Majorana condition. in other literature it sometimes written as

$$\psi_M = C\bar{\psi}_M^T \tag{2.16}$$

where C is the charge conjugation operator defined by

$$C \equiv i\gamma^2\gamma^0 \tag{2.17}$$

using this, the Lagrangian in Eq. (2.8) can be written as

$$\mathcal{L}_M = \bar{\psi}_L\gamma^\mu\partial_\mu\psi_L - \frac{m_M}{2}\left(\psi_L^T C\psi_L + \psi_L^\dagger C\psi_L^*\right) \tag{2.18}$$

Chapter 3

Neutrino Mass and Seesaw Mechanism

With only Standard Model degreese of freedom, the neutrino Yukawa interaction take a $d = 5$ effective form which can be written symbolically as[9]

$$\frac{LLHH}{M} \tag{3.1}$$

There are three ways to write this term which is invariant under $SU(2)_L \times U(1)_Y$ transformation

$$1. \quad \frac{\left(\ell_L^T i\sigma^2 \phi\right) C \left(\phi^T i\sigma^2 \ell_L\right)}{M} \tag{3.2}$$

$$2. \quad \frac{\left(\ell_L^T C i\sigma^2 \vec{\sigma} \ell_L\right) \left(\phi^T i\sigma^2 \vec{\sigma} \phi\right)}{M} \tag{3.3}$$

$$3. \quad \frac{\left(\ell_L^T i\sigma^2 \vec{\sigma} \phi\right) C \left(\phi^T i\sigma^2 \vec{\sigma} \ell_L\right)}{M} \tag{3.4}$$

Where ℓ_L and ϕ are the lepton doublet and higgs respectively. They are defined in the $SU(2)_L \times U(1)_Y$ model as the following[10]

$$\ell_L = \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L \quad , \quad \phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} = \begin{pmatrix} \phi^+ \\ \frac{v+h+i\eta}{\sqrt{2}} \end{pmatrix} \tag{3.5}$$

ϕ^+ and η are the would be goldstone bosons, eaten by W and Z respectively; so that ony h is physical Higgs particle.

We can get an insight on the type of intermediate particle which may give rise to an effective interaction of the type shown in Eq. (3.1) by evaluating the form on Eq. (3.2)-(3.4). In Eq. (3.2) ℓ_L and ϕ combine to form an $SU(2)$ singlet, in Eq. (3.3) both ℓ_L and ϕ form an $SU(2)$ triplet while in Eq. (3.4) ℓ_L and ϕ combine to form an $SU(2)$ triplet. The intermediate particle that gives those form are singlet fermion, triplet scalar, and triplet fermion respectively for Eq. (3.2), (3.3) and (3.4). These three types of intermediate states corresponds to the type of seesaw mechanism which are going to be explained in the next section.

After the breaking $SU(2)_L \times U(1)_Y \to U(1)_Q$ the Higgs will obtain its minimum value and we will get a mass for the left-handed neutrino.

3.1 Seesaw Mechanism

To demonstrate the seesaw mechanism, we give an example on how to generate neutrino mass in the minimal Standard Model. For simplicity, we are going to use only one generation of neutrino.

First, we introduce a new particle ν_R into the model. It is a singlet $SU(2)_L$ with $Y = 0$ so that a Majorana mass term is allowed. The full Lagrangian is the following

$$\mathcal{L} = \mathcal{L}_{SM} + i\bar{\nu}_R\gamma^\mu\partial_\mu\nu_R - \frac{m_S}{2}\nu_R^T C\nu_R + y_S\bar{\ell}_L\tilde{\phi}\nu_R + h.c. \tag{3.6}$$

where y_S is an arbitraty Yukawa coupling parameter. $\tilde{\phi}$ is defined by

$$\tilde{\phi} \equiv i\sigma^2\phi^* \tag{3.7}$$

After symmetry breaking the Higgs will obtain the minimum

$$\langle\phi\rangle = \begin{pmatrix} o \\ v \end{pmatrix} \tag{3.8}$$

So we will have Dirac mass terms which mixes ν_L and ν_R.

$$\mathcal{L}_D = m_D\bar{\nu}_L\nu_R + h.c. \tag{3.9}$$

where m_D is given by

$$m_D = \frac{y_S v}{\sqrt{2}} \tag{3.10}$$

We can sumarize the mass terms by putting it in matrix form

$$\mathcal{L}_{mass} = \begin{pmatrix} \bar{\nu}_L & \bar{\nu}_R \end{pmatrix} \begin{pmatrix} 0 & m_D \\ m_D & m_S \end{pmatrix} \begin{pmatrix} \nu_L \\ \nu_R \end{pmatrix} \tag{3.11}$$

The eigenvalues for the mass matrix on Eq. (3.11) is

$$m_\pm = \frac{m_S}{2} \pm \frac{m_S}{2}\sqrt{1 + \frac{4m_D^2}{m_S^2}} \tag{3.12}$$

(note : the resulting mass will be of Majorana type).

If the new particle is very heavy so that

$$m_S \gg m_D \tag{3.13}$$

Then we will get two states. A heavy one

$$m_+ \simeq m_S \tag{3.14}$$

which consist mostly of ν_R, and a light one

$$m_- \simeq -\frac{m_D^2}{m_S} \tag{3.15}$$

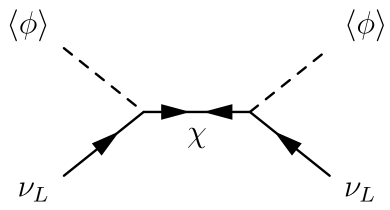
which consist mostly of ν_L.

Thus we have generated a mass for the left-handed neutrino. Its smallness is controlled by the smallness of m_D and the largeness of m_S. We have no information in the Standard Model on these parameters so we must use a well defined theory beyond the Standard Model. Also note that as an immediate consequence of a Majorana neutrino we should have a lepton number violating process.

3.2 Types of seesaw

In the previous section we have demonstrated the seesaw mechanism in the modified Standard Model by adding a singlet ($SU(2)_L$) fermion. That is called the type

I seesaw. In type I seesaw, the neutrino gets a mass as a result of this process



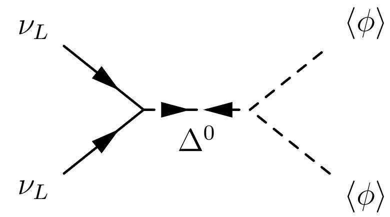
For type II seesaw we add to the Lagrangian a triplet scalar Δ with Y=2. The full Lagrangian will get additional terms given by[11]

$$\delta\mathcal{L} = (D^\mu\vec{\Delta}^*)(D_\mu\vec{\Delta}) + m_\Delta^2\vec{\Delta}^\dagger\vec{\Delta} + y_\Delta^{(1)}(\ell_L^T C i\sigma^2\vec{\sigma}\ell_L)\cdot\vec{\Delta} + \mu_\Delta(\phi^T i\sigma^2\vec{\sigma}\phi)\cdot\vec{\Delta}^* + h.c. \tag{3.16}$$

where

$$\Delta \equiv \begin{pmatrix} \Delta^{++} \\ \Delta^{+} \\ \Delta^{0} \end{pmatrix} \tag{3.17}$$

Neutrinos obtain masses due to interaction with the neutral scalar Δ^0 through the process



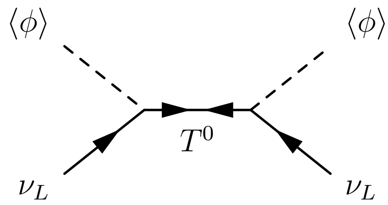
For type III seesaw, we add to the Lagrangian a triplet fermion T_R with Y=0, therefore we will have these new terms on our Lagrangian[11]

$$\delta\mathcal{L} = i\overline{\vec{T}}_R\gamma^\mu D_\mu\vec{T}_R - \frac{m_T}{2}\vec{T}_R^T C\vec{T}_R + y_T\overline{\vec{T}}_R(\phi^T i\sigma^2\vec{\sigma}\ell_L) + h.c. \tag{3.18}$$

where

$$T = \begin{pmatrix} T^3 \\ T^2 \\ T^1 \end{pmatrix} \tag{3.19}$$

and neutrinos obtain their masses from this process



3.3 Lepton mixing with Majorana neutrinos

We now generalize our discussion for the case of more generations. Just as in the quark sector, there will be generation mixing in the leptonic sector. However, the fact that we are using majorana neutrinos gives us a rather different mixing matrix than the regular CKM matrix for the quarks.

We start by postulating three heavy majorana particles $\nu_R^{1,2,3}$. In that case, our Lagrangian becomes

$$\mathcal{L} = \mathcal{L}_{SM} + i\nu_R^i \gamma^\mu \partial_\mu \nu_R^i + M_S^{ii} \nu_R^{iT} C \nu_R^i + y_S^{ij} \bar{\ell}^i \tilde{\phi} \nu_R^j \tag{3.20}$$

We put the Higgs VEV as before so that we acquire neutrino mass terms

$$\begin{aligned} \mathcal{L}_M &= \mathbf{m}_D \bar{\vec{\nu}}_L \vec{\nu}_R + \mathbf{M}_S \vec{\nu}_R^T C \vec{\nu}_R + h.c. \\ &= \begin{pmatrix} \bar{\vec{\nu}}_L & \bar{\vec{\nu}}_R \end{pmatrix} \begin{pmatrix} 0 & \mathbf{m}_D \\ \mathbf{m}_D^T & \mathbf{M}_S \end{pmatrix} \begin{pmatrix} \vec{\nu}_L \\ \vec{\nu}_R \end{pmatrix} \end{aligned} \tag{3.21}$$

We have chosen $\mathbf{M}_S$ to be diagonal which can be done without loss of generality since we can always rotate the $\vec{\nu}_R$ states. $\mathbf{m}_D$ and $\mathbf{M}_S$ are 3×3 matrix given by

$$\begin{aligned} m_D^{ij} &= \frac{y^{ij} v}{\sqrt{2}} \\ M^{ij} &= M_S^i \delta^{ij} \end{aligned} \tag{3.22}$$

and $\vec{\nu}_L$ and $\vec{\nu}_R$ are 3-component vector defined by

$$\nu_L \equiv \begin{pmatrix} \nu^1 \\ \nu^2 \\ \nu^3 \end{pmatrix}_L \quad ; \quad \nu_R \equiv \begin{pmatrix} \nu^1 \\ \nu^2 \\ \nu^3 \end{pmatrix}_R \tag{3.23}$$

Here the neutrinos are in mass eigenstates (number in the index denote mass eigenstates). After integrating out ν_R we will obtain the light neutrino mass terms

$$m_\nu^{ij} \nu_L^{iT} C \nu_L^j \tag{3.24}$$

where $\mathbf{m}_\nu$ is a symmetric matrix given by

$$\mathbf{m}_\nu = -\mathbf{m}_D^T \frac{1}{\mathbf{M}_S} \mathbf{m}_D \tag{3.25}$$

Working in the flavour basis in which the charged lepton mass matrix and gauge interactions are flavour diagonal, $\mathbf{m}_\nu$ are diagonalized by the mixing matrix U according to

$$U^T m_\nu U = \mathrm{diag}(m_1, m_2, m_3) \tag{3.26}$$

U also relates the neutrino flavour and mass eigenstates by

$$\begin{pmatrix} \nu^e \\ \nu^\mu \\ \nu^\tau \end{pmatrix} = U \begin{pmatrix} \nu^1 \\ \nu^2 \\ \nu^3 \end{pmatrix} \tag{3.27}$$

The mixing matrix U is unitary so that it will have 6 phases. 3 of these phases can be removed by rotating the lepton states. The neutrino states however can't be rotated because it has Majorana mass forms

$$m_\nu \nu_L^T C \nu_L \tag{3.28}$$

Therefore in the end we will have 3 physical phases on the lepton mixing matrix U. The matrix U can be parametrized using the usual CKM-like matrix multiplied by an additional phase matrix

$$U \equiv V_{CKM} K \tag{3.29}$$

where

$$K \equiv \begin{pmatrix} e^{-i\phi/2} & 0 & 0 \\ 0 & e^{-i\phi'/2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.30}$$

and V_{CKM} can be chosen by the usual parametrization

$$V_{CKM} = \begin{pmatrix} c_{13}c_{12} & c_{13}s_{12} & s_{13}e^{-i\delta} \\ -c_{23}s_{12} - s_{23}s_{13}c_{12}e^{i\delta} & c_{23}c_{12} - s_{23}s_{13}s_{12}e^{i\delta} & s_{23}c_{13} \\ s_{23}s_{12} - c_{23}s_{13}c_{12}e^{i\delta} & -s_{23}c_{12} - c_{23}s_{13}s_{12}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \tag{3.31}$$

Chapter 4

The $SU(5)$ Grand Unified Theory

The Glashow-Weinberg-Salam $SU(3)_C \times SU(2)_L \times U(1)_Y$ model has three gauge couplings of different magnitudes and fermions assigned according to convenience, rather than any principle. It also has many parameters that are adjusted arbitrarily to fit different observations such as masses and mixings of quarks, etc. A logical next step to consider is a higher symmetry that unifies all three couplings and also simultaneously offers the possibility of relating the different parameters so that one has a more satisfactory theory than the standard model.

In this work we are going to work on the $SU(5)$ group which first proposed by Georgi and Glashow. Using this model we will not only able to explain the quantization of electric charge but also leads to unification of all coupling constants.

4.1 SU(5) gauge bosons

The generator of $SU(5)$ is defined as[12]

$$T_i = \frac{1}{2}\lambda_i \tag{4.1}$$

Where $i = 1 \ldots 24$ and λ_i are given in the appendix. We can then construct an adjoint representation for gauge bosons.

$$\sum_{i=1}^{24} \frac{1}{\sqrt{2}} \lambda_i A_i = \begin{pmatrix} \frac{1}{\sqrt{2}}\vec{\lambda}^{\{8\}} \cdot \vec{A}_{\{8\}} + \sqrt{\frac{2}{15}} A_{24} & & & \overline{X}_1 & \overline{Y}_1 \\ & & & \overline{X}_2 & \overline{Y}_2 \\ & & & \overline{X}_3 & \overline{Y}_3 \\ X_1 & X_2 & X_3 & \frac{1}{\sqrt{2}}\vec{\sigma} \cdot \vec{W} - \sqrt{\frac{3}{10}} A_{24} & \\ Y_1 & Y_2 & Y_3 & & \end{pmatrix} \tag{4.2}$$

The X and Y bosons are new gauge bosons with electric charge 4/3 and 1/3 respectively which defined by

$$X_i = \frac{A_{5+4i} + iA_{6+4i}}{\sqrt{2}}, \quad Y_i = \frac{A_{7+4i} + iA_{8+4i}}{\sqrt{2}}, \quad i = 1, \ldots, 3 \tag{4.3}$$

They are introduced in SU(5) to facilitate direct interaction between quarks and leptons.

4.2 Fermions

There are 15 Weyl fields, we put them on the {5}- and {10}- dimensional representations given by

$$5_F \equiv \begin{pmatrix} d_r \\ d_g \\ d_b \\ e^+ \\ v^C \end{pmatrix}_R, \quad 10_F \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & u_b^C & -u_g^C & u_r & d_r \\ -u_b^C & 0 & u_r^C & u_g & d_g \\ u_g^C & -u_r^C & 0 & u_b & d_b \\ -u_r & -u_g & -u_b & 0 & e^+ \\ -d_r & -d_g & -d_b & -e^+ & 0 \end{pmatrix}_L \tag{4.4}$$

(Note: $\psi_L^C \equiv C\overline{\psi}_R^T$)

Moreover we see that this theory explains charge quantization, i.e. it relates quark and lepton charges. From Eq. (4.4)

$$Q(d) = -\frac{1}{3} Q(e^+) = -\frac{1}{3} \tag{4.5}$$

and

$$Q(u) = Q(d) + 1 = \frac{2}{3} \tag{4.6}$$

The interaction between fermions and gauge bosons is given by

$$\mathcal{L}_f = i\overline{5}_F\gamma^\mu D_\mu 5_F - iTr\overline{10}_F\gamma^\mu D_\mu 10_F \tag{4.7}$$

where the covariant derivatives are given by

$$D_\mu 5_F = \partial_\mu 5_F - igA_\mu 5F \tag{4.8}$$

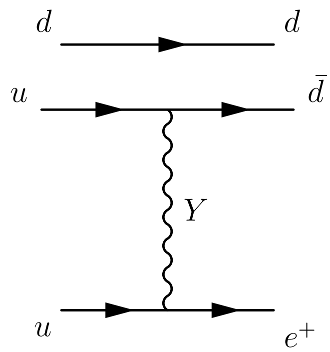
and

$$D_\mu 10_F = \partial_\mu 10_F - ig(A_\mu 10_F + 10_F A_\mu^T) \tag{4.9}$$

These interactions will include the QCD and electroweak interactions with $g_S = g_W = g$. The new interactions coming from X and Y bosons are

$$\begin{aligned}\mathcal{L}(X,Y) &= \frac{g}{\sqrt{2}}\overline{X}_\mu^\alpha\left[\overline{d}_{\alpha R}\gamma^\mu e_R^+ + \overline{d}_{\alpha L}\gamma^\mu e_L^+ + \epsilon_{\alpha\beta\gamma}\overline{u}_L^{C\gamma}\gamma^\mu u_{\beta L}\right] \\ &+ \frac{g}{\sqrt{2}}\overline{Y}_\mu^\alpha\left[-\overline{d}_{\alpha R}\gamma^\mu \nu_R^C + \overline{u}_{\alpha L}\gamma^\mu e_L^+ + \epsilon_{\alpha\beta\gamma}\overline{u}_L^{C\gamma}\gamma^\mu d_{\beta L}\right] \end{aligned} \tag{4.10}$$

These new interactions will give rise to a proton decay. The decay may occur through the mode $p \to e^+\pi^0$ via the diagram



The decay rate for this process can be evaluated by analogy with the weak decay $n \to p + e + \bar{\nu}$, thus we will obtain

$$\Gamma_p \simeq \frac{g^4}{M_Y^4} m_p^5 \tag{4.11}$$

The experiment gives us the limit $\tau_{exp} > 10^{33} yr$ hence we get $M_Y > 10^{15.5} GeV$.

4.3 Higgs and Symmetry Breaking

We will have two steps of breaking

$$\begin{aligned} SU(5) &\rightarrow SU(3)_C \times SU(2)_L \times U(1)_Y \\ &\rightarrow SU(3)_C \times U(1)_Q \end{aligned} \tag{4.12}$$

The first breaking is achieved by {24}-dimensional representation Higgs given by

$$24_H = \begin{pmatrix} \frac{1}{\sqrt{2}}\Sigma_8 + \sqrt{\frac{2}{15}}\Sigma_{24} & \overline{\Sigma}_X & \overline{\Sigma}_Y \\ \Sigma_X & \frac{1}{\sqrt{2}}\Sigma_3 + \sqrt{\frac{3}{10}}\Sigma_{24} & \Sigma^+ \\ \Sigma_Y & \Sigma^- & -\frac{1}{\sqrt{2}}\Sigma_3 + \sqrt{\frac{3}{10}}\Sigma_{24} \end{pmatrix} \tag{4.13}$$

Where Σ_8, Σ_3, $\Sigma_\pm$ and Σ_{24} are the analog of gluons, W^3, $W^\pm$ and B bosons respectively and Σ_X, Σ_Y for the new X and Y bosons. Moreover the second breaking achieved by {5}-dimensional representation Higgs

$$5_H = \begin{pmatrix} h^r \\ h^g \\ h^b \\ \phi^+ \\ \phi^0 \end{pmatrix} \tag{4.14}$$

where $h^{r,g,b}$ are new color triplet scalars and ϕ^+, ϕ^0 are the usual SM Higgs.

The full potential for this Higgs is given by

$$\begin{aligned} V &= -\frac{\mu_{24}^2}{2} Tr24_H^2 + \frac{a}{4}(Tr24_H^2)^2 + \frac{b}{2}Tr24_H^4 \\ &- \frac{\mu_5^2}{2}5_H^\dagger 5_H + \frac{\lambda}{4}(5_H^\dagger 5_H)^2 \\ &+ \alpha 5_H^\dagger 5_H Tr24_H^2 - \beta 5_H^\dagger 24_H^2 5_H \end{aligned} \tag{4.15}$$

The minimum of this potential is given by

$$\langle 24_H \rangle = v_X \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & -3/2-\epsilon/2 & \\ & & & & -3/2+\epsilon/2 \end{pmatrix} \tag{4.16}$$

and

$$\langle 5_H \rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ v_W/\sqrt{2} \end{pmatrix} \tag{4.17}$$

Substituting Eqs. (4.16) and (4.17) into the potential in Eq. (4.15), we get

$$\begin{aligned} V_{min} &= -\frac{1}{2}\mu_{24}^2 v_X^2 \left(\frac{15}{2}+\frac{\epsilon^2}{2}\right) + \frac{a}{4}v_X^4 \left(\frac{15}{2}+\frac{\epsilon^2}{2}\right)^2 + \frac{b}{2}v_X^4 \left(\frac{105}{8}+\frac{54}{8}\epsilon^2+\dots\right) \\ &\quad -\frac{\mu_5^2}{4}v_W^2 + \frac{\lambda}{16}v_W^4 + \frac{\alpha}{2}v_W^2 v_X^2 \left(\frac{15}{2}+\frac{\epsilon^2}{2}\right) + \beta\frac{v_W^2 v_X^2}{8}(3-\epsilon)^2 \end{aligned} \tag{4.18}$$

The conditions for minimum give the following relations

$$\begin{aligned} \epsilon &\simeq \frac{2\beta}{20b}\left(\frac{v_W}{v_X}\right)^2 \\ \mu_{24}^2 &= \frac{15}{2}av_X^2 + \frac{7}{2}bv_X^2 + \alpha v_W^2 + \frac{3}{10}\beta v_W^2 + O(\epsilon v_W^2) \\ \mu_5^2 &= \frac{1}{2}\lambda v_W^2 + 15\alpha v_X^2 + \frac{9}{2}\beta v_X^2 - 3\epsilon\beta v_X^2 + O(\epsilon v_W^2) \end{aligned} \tag{4.19}$$

Notice on the third line of Eq. (4.19), the LHS (μ_5^2) is of the order of $10^4 GeV^2$ while the RHS contains the terms with v_X which is of the order $10^{15.5} GeV$. This requires us to have a very high precission on fixing the α and β parameters (i.e. we should have $(15\alpha + \frac{9}{2}\beta) \leq 10^{-27}$). This is called the hierarcy problem of Grand Unified Theory.

We can obtain the mass of each Higgs particle by expanding 24_H around its minimum. After some calculation we will obtain the result as given below

$$\begin{aligned} m^2(\Sigma_8) &\simeq \frac{5}{2}bv_X^2 \\ m^2(\Sigma_3) = m^2(\Sigma_\pm) &\simeq 10bv_X^2 \\ m^2(\Sigma_0) &\simeq (15a + 7b)v_X^2 \\ m^2(\Sigma_X) = m^2(\Sigma_Y) &= 0 \end{aligned}$$

$$\begin{aligned} m^2(h) &= \frac{5}{2}\beta v_X^2 \\ m^2(\phi^\pm) = m^2(\mathrm{Im}(\phi^0)) &= 0 \\ m^2(\mathrm{Re}(\phi^0)) &= \lambda v_W^2 \end{aligned} \tag{4.20}$$

In order for the masses to have a positive value we will require $b > 0$, $\lambda > 0$, $15a+7b > 0$ and $\beta > 0$. Also notice from Eq. (4.20) that Σ_X and Σ_Y are massless, these are the Goldstone bosons which will be "eaten" by X and Y bosons to make them massive.

4.4 Yukawa coupling and fermion masses

In the $SU(5)$ model, the fermion masses originate through the Yukawa couplings of fermions with the light Higgs

$$\mathcal{L}_Y = f_d \bar{5}_F^i 10_F^{ij} 5_H^{j*} + f_u \frac{1}{2}\epsilon_{ijklm}(10_F^T)^{ij} C 10_F^{kl} 5_H^m \tag{4.21}$$

where C is the Dirac conjugation matrix, and f_u is a symmetric matrix. After putting the minimum of Φ we will get terms for fermionic mass

$$\begin{aligned} \mathcal{L}_m &= f_d v_W(\bar{d}_R d_L + \bar{e}_R^+ e_L^+) - f_u v_W (u^c)_L^T C u_L + h.c. \\ &= -[f_d v_W(\bar{d}d + \bar{e}e) - f_u v_W \bar{u}u] \end{aligned} \tag{4.22}$$

It is interesting to see here that the mass of down quark and electron are the same. This shows an additional symmetry between down quark and electron. The new symmetry is SU(4) which is obvious since the Higgs VEV that we used to generate the fermion masses (5_H) doesn't break the SU(4) symmetry. Still, this mass only valid in the GUT scale and we need to run it down to M_W if we want to compare it with experiment.

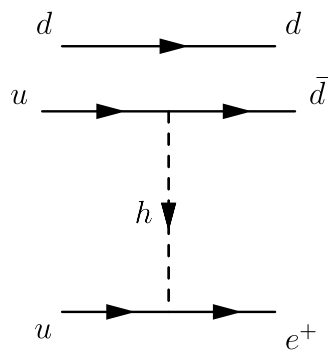
Beside the mass terms, from Eq. (4.21) we will also have new interactions with the color triplet h_α. Its interaction are

$$\mathcal{L}_h = f_d \bar{5}_{Fi} 10_F^{i\alpha} h_\alpha^* + f_u \epsilon_{ijkl\alpha} (10_F^T)^{ij} C 10_F^{kl} h^\alpha + h.c. \quad (4.23)$$

which gives

$$\begin{aligned} \mathcal{L}_h &= f_d(\epsilon^{\alpha\beta\gamma} u_{R\beta}^T C d_R^\gamma + \bar{d}_L^\alpha e_R^+ + \bar{d}_L^\alpha C \bar{\nu}_L) \\ &+ f_u(\epsilon^{\alpha\beta\gamma} u_{L\beta}^T C d_L^\gamma + \bar{u}_R^\alpha e_L^+) h^\alpha + h.c. \end{aligned} \quad (4.24)$$

This interactions with color triplets may also give rise to proton decay through



The amplitude for this process will be proportional to the small Yukawa couplings. Therefore the limit on m_h is less strict then the one on M_X:$m_h \geq 10^{-12} GeV$.

4.5 Coupling unification on high scale

The generic formula for coupling constant of the gauge group G at μ scale with respect to μ_0 is given by

$$\frac{1}{\alpha_G(\mu)} - \frac{1}{\alpha_G(\mu_0)} = \frac{b_G}{2\pi} ln \frac{\mu}{\mu_0} \quad (4.25)$$

The coefficient b_G is defined as the following

$$b_G = \frac{11}{3}T_{GB} - \frac{2}{3}T_F - \frac{1}{3}T_H \tag{4.26}$$

where $T = T_G^2$ (T_G is the generator of group G). Moreover T_{GB}, T_F, T_H correspond to gauge bosons, fermions and scalar (higgs) particle respectively. Thus we will have $T = \frac{1}{2}$ for a fundamental representation and $T = N$ for an adjoint representation of size N. We will have for the $SU(3)_C$, $SU(2)_L$ and $U(1)$ respectively

$$\begin{aligned} b_3 &= \frac{33}{3} - 4 = 7 \\ b_2 &= \frac{22}{3} - 4 - \frac{1}{6} = \frac{19}{6} \\ b_1 &= \frac{3}{5}b_Y = -4 - \frac{1}{10} = -\frac{41}{10} \end{aligned} \tag{4.27}$$

The factor $\frac{3}{5}$ is the normalization factor added since Y^2 normalized differently with λ^2.[1]

The coupling constants can be evaluated using all this relations, the result is shown on Fig.(4.5). It is easy to check that α_1 and α_2 meet at $\mu \simeq 10^{14} GeV$ and α_2 and α_3 meet at $\mu \simeq 10^{16} GeV$. In other words the three couplings don't meet at the same scale. This is the main problem of the $SU(5)$ model of Grand Unification. If we forced the coupling to meet at some high scale then we will find ratio between coupling constant (i.e. $sin^2\theta_W$) on our current scale (m_W) to be different than what measured in the experiment.

To tackle this problem we are going to add new particles to the model to slow down the decrease of weak coupling constant in such a way that all the coupling meet together at one point. As we will see later, the constraint of GUT scale from

[1] as an example, take the 5_F representation. For that group, the sum of all Y^2 is

$$\begin{aligned} \sum_i Y_i^2 &= \frac{4}{9} + \frac{4}{9} + \frac{4}{9} + 1 + 1 \\ &= \frac{10}{3} \end{aligned} \tag{4.28}$$

While $\lambda^2 = 2$, thus to make it equal we should multiply $b_Y = Y^2$ by $\frac{3}{5}$

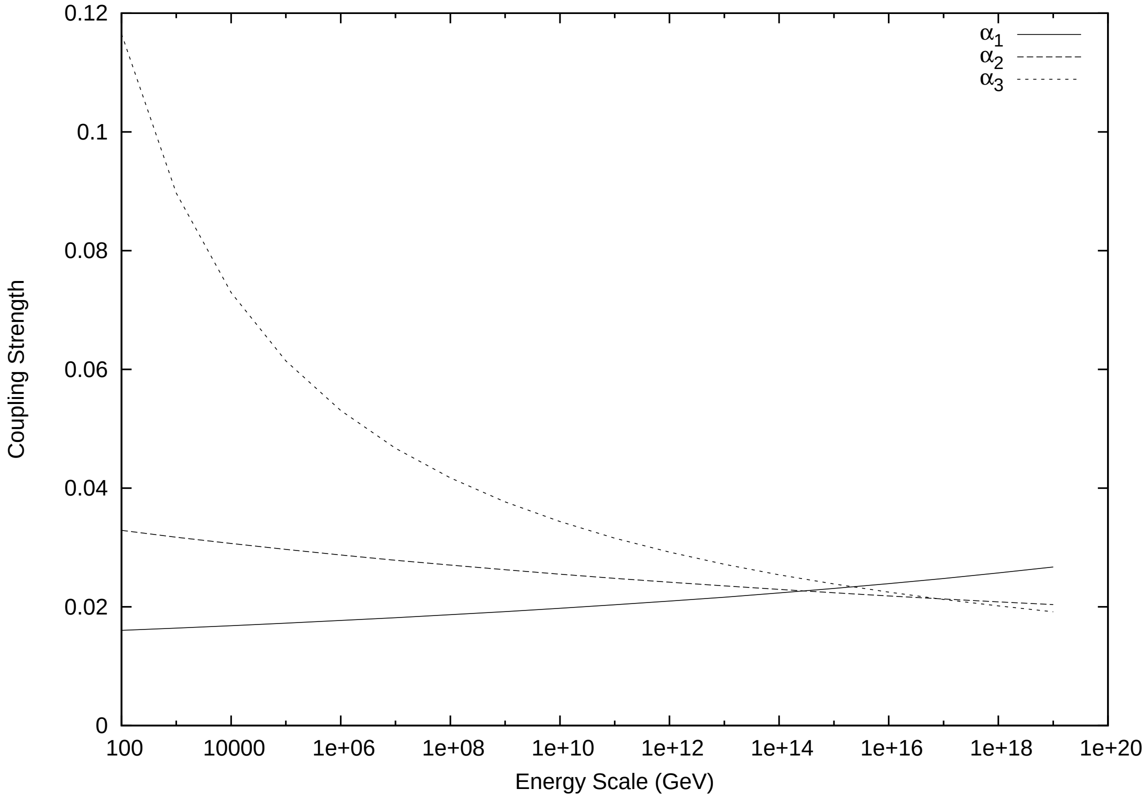


Figure 4.1: The evolution of coupling constants with energy. With respect to Eq. (4.25) we have put $\mu_0 = M_W$, $\alpha_3(\mu_0) = 0.12$, $\alpha_2(\mu_0) = 0.033$ and $\alpha_1(\mu_0) = 0.016$. Notice that the three coupling constants does not meet together at the same point.

proton decay experiment will force us to make one of the new particles to be light which we hope can be detected in the near future.

Chapter 5

Implementing Seesaw on the $SU(5)$ GUT

In this chapter we would like to implement the seesaw mechanism in the SU(5) model to enable us to have massive neutrinos. From the solar and atmospheric oscillation data, we know that at least two neutrinos are massive. Therefore we will need at least two heavy right handed neutrinos. In this work we are going to use a singlet and a triplet fermion as the heavy particles. To do so we will need to add new set of particles to the model. We will see later that the addition of singlet and triplet fermion may help us solve the problem concerning the coupling unification on SU(5) model.

5.1 The adjoint fermion

We will put the new singlet and triplet fermion inside the adjoint representation 24_F which is given by

$$24_F = \begin{pmatrix} \frac{1}{\sqrt{2}}\sigma_8 + \sqrt{\frac{2}{15}}S \cdot \mathbf{1} & \sigma_{(3,2)} \\ \bar{\sigma}_{(3,2)} & \frac{1}{\sqrt{2}}\vec{\sigma} \cdot \vec{T} + \sqrt{\frac{3}{10}}S \cdot \mathbf{1} \end{pmatrix} \tag{5.1}$$

the Eq. (5.1) is to be read as a 5×5 matrix, its form is the fermion equivalent of the $SU(5)$ gauge boson matrix.

The full Lagrangian for 24_F is given by

$$\mathcal{L}_{24_F} = \mathcal{L}_{kin} + \mathcal{L}_Y + \mathcal{L}_F \tag{5.2}$$

where

$$\mathcal{L}_{kin} = i\overline{24}_F \gamma^\mu D_\mu 24_F \tag{5.3}$$

$$\begin{aligned}\mathcal{L}_Y = {} & y_0^i \overline{5}_F^i 24_F 5_H \\ & + \frac{1}{\Lambda}\overline{5}_F^i (y_1^i 24_F 24_H + y_2^i 24_H 24_F + y_3^i Tr 24_F 24_H) 5_H + h.c.\end{aligned} \tag{5.4}$$

$$\begin{aligned}\mathcal{L}_F = {} & m_F Tr 24_F^2 + \lambda_F Tr 24_F^2 24_H \\ & + \frac{1}{\Lambda}(a_1 Tr 24_F^2 Tr 24_H^2 + a_2 (Tr 24_F 24_H)^2 \\ & + a_3 Tr 24_F^2 Tr 24_H^2 + a_4 Tr 24_F 24_H 24_F 24_H)\end{aligned} \tag{5.5}$$

In the previous equations we have used f^{ijk} as the structure factor[1] and Λ as cutoff parameter which we can take as $\Lambda = M_{\text{Planck}}$. Moreover, the higher dimensional terms are put by hand in order to split the mass of the triplet fermion (more detail on this later).

After the first breaking to the Standard Model by inserting $\langle 24_H \rangle$, we will obtain the masses of new fermions

$$m_S^F = m_F - \frac{\lambda_F M_X}{\sqrt{30}} + \frac{M_X^2}{\Lambda}\left[a_1 + a_2 + \frac{7}{30}(a_3 + a_4)\right] \tag{5.7}$$

$$m_T^F = m_F - \frac{3\lambda_F M_X}{\sqrt{30}} + \frac{M_X^2}{\Lambda}\left[a_1 + \frac{3}{10}(a_3 + a_4)\right] \tag{5.8}$$

$$m_8^F = m_F + \frac{2\lambda_F M_X}{\sqrt{30}} + \frac{M_X^2}{\Lambda}\left[a_1 + \frac{2}{15}(a_3 + a_4)\right] \tag{5.9}$$

$$m_{(3,2)}^F = m_F - \frac{\lambda_F M_X}{2\sqrt{30}} + \frac{M_X^2}{\Lambda}\left[a_1 + \frac{(13a_3 - 12a_4)}{60}\right] \tag{5.10}$$

[1]given from

$$[\lambda^i, \lambda^j] = i f^{ijk} \lambda^k \tag{5.6}$$

5.2 The light triplet fermion

We have seen in the previous chapter that the failure of SU(5) model is due to the fact that the three couplings on the Standard Model do not come altogether to one point of "Grand Unification" on high scale. We will show now that to fix this problem we will require the triplet fermion on 24_F to be as light as possible.

We will redo the evaluation of running coupling constant we did before, except this time we add the contribution from all the new particles. Using the Eq. (4.25) we will obtain the following relation

$$2\pi\left(\frac{1}{\alpha_1(M_W)} - \frac{1}{\alpha_U}\right) = \frac{41}{10}ln\frac{M_X}{M_W} + \frac{10}{3}ln\frac{M_X}{m^F_{(3,2)}} + \frac{1}{15}ln\frac{M_X}{m^h} \tag{5.11}$$

$$\begin{aligned} 2\pi\left(\frac{1}{\alpha_2(M_W)} - \frac{1}{\alpha_U}\right) &= -\frac{3}{2}ln\frac{M_X}{M_W} - \frac{4}{3}ln\frac{m^F_T}{M_W} \\ &\quad -\frac{1}{3}ln\frac{m^\Sigma_3}{M_W} + 2ln\frac{M_X}{m^F_{(3,2)}} \end{aligned} \tag{5.12}$$

$$\begin{aligned} 2\pi\left(\frac{1}{\alpha_3(M_W)} - \frac{1}{\alpha_U}\right) &= -\frac{9}{2}ln\frac{M_X}{M_W} - 2ln\frac{m^F_8}{M_W} - \frac{1}{2}ln\frac{m^\Sigma_8}{M_Z} \\ &\quad +\frac{4}{3}ln\frac{M_X}{m^F_{(3,2)}} + \frac{1}{6}ln\frac{M_X}{m^h} \end{aligned} \tag{5.13}$$

where m^F's, m^Σ's and m^h's correspond to mass of 24_F, 24_H and color triplet from 5_H respectively (refer to Eq. (4.13) and Eq. (4.14) for the notation). In the previous equation we have run the coupling from M_W scale to M_X scale through the following step

$$M_W \rightarrow m^F_T \rightarrow m^\Sigma_3 \rightarrow m^F_8 \rightarrow m^\Sigma_8 \rightarrow m^F_{(3,2)} \rightarrow m^h \rightarrow M_X \tag{5.14}$$

From the above relations, a straightforward computation will give

$$exp[30\pi(\alpha_1^{-1} - \alpha_2^{-1})(M_W)] = \left(\frac{M_X}{M_W}\right)^{84}\left(\frac{(m^F_T)^4 m^\Sigma_3}{M_W^5}\right)^5\left(\frac{M_X}{m^F_{(3,2)}}\right)^{20}\left(\frac{M_X}{m^h}\right) \tag{5.15}$$

$$exp[20\pi(\alpha_1^{-1} - \alpha_3^{-1})(M_W)] = \left(\frac{M_X}{M_W}\right)^{86}\left(\frac{(m^F_8)^4 m^\Sigma_8}{M_W^5}\right)^5\left(\frac{M_X}{m^F_{(3,2)}}\right)^{20}\left(\frac{M_X}{m^h}\right)^{-1} \tag{5.16}$$

It is clear that to slow down the decrease of α_2 and increase the meeting point of α_1 and α_2 we need to make m_T^F as small as possible while keeping m^h and $m_{(3,2)}^F$ as big as possible. To illustrate this point, take $m_T^F = m_3^\Sigma = M_W$ and $m^h = M_X$. With $\alpha_1^{-1}(M_W) = 59, \alpha_2^{-1}(M_W) = 29.57$ and $\alpha_3^{-1}(M_W) = 8.55$ we will get $M_X \approx 10^{15.5} GeV$. Increasing the masses $m_3^{F,\Sigma}$ reduces M_X dangerously making the proton decay too fast.

Additionaly, if we want to make the triplet fermion lighter than the rest of the fermions on 24_F we will need to add the higher dimensional terms on its interaction terms with the 24_H Higgs. This in turn give an upper limit for $m_{(3,2)}^F$

$$m_{(3,2)}^F \lesssim \frac{M_X^2}{\Lambda} \tag{5.17}$$

To conclude this part we stress here that the triplet fermion should be very light close to M_W. This make this model quite interesting, since we can directly probe it on the future collider experiment such as LHC.

5.3 Seesaw mechanism with the new singlet and triplet fermions

We will now go through the detail of generating massive neutrinos throught seesaw mechanism on this model. To simplify our problem we will focus only on the part with leptons, singlet and triplet fermion. For the triplets we are using the following 4-vector notation

$$T^- = \begin{pmatrix} T_L^- \\ T_R^- \end{pmatrix} \quad ; \quad T^3 = \begin{pmatrix} T_L^3 \\ T_R^3 \end{pmatrix} \tag{5.18}$$

and T^+ is related to T^- by the relation $T^+ = C\bar{T}^{-T}$.

5.3.1 The Lagrangian

The kinetic part of the Lagrangian is

$$\begin{aligned}\mathcal{L}_{kin} &= i\bar{\ell}_L\gamma^\mu\left(\partial_\mu + i\frac{g}{2}\sigma^i A^i_\mu - i\frac{g'}{2}B_\mu\right)\ell_L \\ &\quad + i\bar{T}^i_L\gamma^\mu\left(\partial_\mu T^i_L + g\epsilon^{ijk}A^j_\mu T^k_L\right) \\ &\quad + i\bar{S}_L\gamma^\mu\partial_\mu S_L + i\bar{e}_R\gamma^\mu\left(\partial_\mu + ig'B_\mu\right)e_R \end{aligned}\tag{5.19}$$

where index i on T^i_L is $i = 1, 2, 3$.

The triplet and singlet fermions have the majorana mass term given by

$$\begin{aligned}\mathcal{L}_m &= -\frac{m^F_T}{2}\left(2T^{+T}_L C T^-_L + T^{3T}_L C T^3_L\right) - \frac{m^F_S}{2}S^T_L C S_L + h.c. \\ &= -\frac{m^F_T}{2}(2\bar{T}^-_R T^-_L + T^{3T}_L C T^3_L) - \frac{m^F_S}{2}S^T_L C S_L + h.c.\end{aligned}\tag{5.20}$$

where we have defined

$$T^\pm = \frac{T^1 \mp iT^2}{\sqrt{2}}\tag{5.21}$$

and we used the fact that $T^+_R = C\bar{T}^{-T}_L$.

Finally the Yukawa terms are given by (we have chosen the basis where the Dirac Yukawa matrix between ℓ_L and e_R is diagonal)

$$\mathcal{L}_Y = -y^i_E\bar{\ell}^i_L\phi e^i_R + y^i_T\vec{T}^T_L C(\phi^T i\sigma^2\vec{\sigma}\ell_L) + y^i_S S^T_L C(\phi^T i\sigma^2\ell_L) + h.c.\tag{5.22}$$

5.3.2 Gauge and Yukawa interactions

When we break $SU(2)_L \times U(1)_Y \to U(1)_Q$ we substitute ϕ with its vev, hence Eq. (5.22) becomes

$$\mathcal{L}_Y \to -\frac{v+h}{\sqrt{2}}\left[y^i_E\bar{e}_R e_L + y^i_T(\sqrt{2}\bar{e}_L T^-_R + \nu^{iT}_L C T^3_L) + y^i_S\nu^{iT}_L C S_L\right] + h.c.\tag{5.23}$$

Next we should consider the gauge interaction of the fermion after defining ($c_W \equiv \cos\theta_W, s_W \equiv \sin\theta_W$)

$$\begin{aligned}B_\mu = c_W A_\mu - s_W Z_\mu \quad &, \quad A^1_\mu = \frac{W^-_\mu + W^+_\mu}{\sqrt{2}} \\ A^3_\mu = s_W A_\mu + c_W Z_\mu \quad &, \quad A^2_\mu = \frac{W^-_\mu - W^+_\mu}{i\sqrt{2}}\end{aligned}\tag{5.24}$$

The interaction on Eq. (5.19) becomes

$$\begin{aligned}
\mathcal{L}_{gauge} \quad &= eA_\mu \quad \left(\bar{e}^i\gamma^\mu e^i + \overline{T^-}\gamma^\mu T^-\right) \\
&-\tfrac{eZ_\mu}{2s_Wc_W} \quad \left(\bar{\nu}^i_L\gamma^\mu\nu^i_L + (2s^2_W-1)\bar{e}^i_L\gamma^\mu e^i_L + 2s^2_W\bar{e}^i_R\gamma^\mu e^i_R\right. \\
&\qquad\qquad \left.-2c^2_W\overline{T^-}\gamma^\mu T^-\right) \\
&-\tfrac{e}{s_W}W^+_\mu \quad \left(\frac{1}{\sqrt{2}}\bar{\nu}^i_L\gamma^\mu e^i_L + \overline{T^3}\gamma^\mu T^-\right) \\
&-\tfrac{e}{s_W}W^-_\mu \quad \left(\frac{1}{\sqrt{2}}\bar{e}^i_L\gamma^\mu\nu^i_L + \overline{T^-}\gamma^\mu T^0\right) \qquad (5.25)
\end{aligned}$$

Notice that in Eq. (5.25) the triplet weak current is vectorlike, this is because T^+ and T^- gives the same contribution to the axial vector current which in turn cancel its axial vector part since $T^+_L \sim T^-_R$.

5.3.3 Neutral mass eigenbasis

The mass terms for the neutral particles are

$$-\frac{1}{2}\begin{pmatrix} \nu^{iT}_L & T^{3T}_L & S^T_L \end{pmatrix} C \begin{pmatrix} 0 & vy^i_T/\sqrt{2} & vy^i_S/\sqrt{2} \\ vy^i_T/\sqrt{2} & m_T & 0 \\ vy^i_S/\sqrt{2} & 0 & m_S \end{pmatrix} \begin{pmatrix} \nu^i_L \\ T^3_L \\ S_L \end{pmatrix} + h.c. \quad (5.26)$$

Note that the hermitian conjugate of Eq. (5.26) contains the right-handed part, i.e. terms with $m_S S^T_R C S_R$, etc. The symmetric complex 5×5 mass matrix can be diagonalized by a unitary transformation

$$\begin{pmatrix} \nu^i \\ T^3 \\ S \end{pmatrix} \to U \begin{pmatrix} \nu^i \\ T^3 \\ S \end{pmatrix} \quad (5.27)$$

Intuitively we can think that the triplet addition give rise to lepton number violation process whose decay rate is governed by either $y_{T,S}$ and $m_{T,S}$. The rarity of such process can means either $y_{T,S}$ very small or $m_{T,S}$ very large. In either case it is logical to consider $|y^i_{T,S}v| \ll m_{T,S}$. We then construct a unitary matrix U up to leading order for $|y^i_{T,S}v| \ll m_{T,S}$.

$$U = \begin{pmatrix} 1_{3\times3} & \epsilon_T & \epsilon_S \\ -\epsilon_T^T & 1 & 0 \\ -\epsilon_S^T & 0 & 1 \end{pmatrix} \quad , \quad \epsilon^i_X = \frac{v y^i_X}{\sqrt{2} m_X} \tag{5.28}$$

After this transformation the mass terms of the neutral fields become approximately

$$-\frac{1}{2} m_T T_L^{3T} C T_L^3 - \frac{1}{2} m_S S_L^T C S_L - \frac{1}{2} m^\nu_{ij} \nu_L^{iT} C \nu_L^j + h.c. \tag{5.29}$$

with

$$\begin{aligned} m^\nu_{ij} &= -m_T \epsilon^i_T \epsilon^j_T - m_S \epsilon^i_S \epsilon^j_S \\ &= -\left(\frac{y^i_T y^j_T}{m_T} + \frac{y^i_T y^j_T}{m_T} \right) \frac{v^2}{\sqrt{2}} \end{aligned} \tag{5.30}$$

which is hybrid of Type I and Type III seesaw.

5.3.4 Charged mass eigenbasis

The mass matrix for charged fields is

$$-\begin{pmatrix} \bar{e}^i_R & \bar{T}^-_R \end{pmatrix} \begin{pmatrix} m^i_e \delta_{ij} & 0 \\ v y^j_T & m_T \end{pmatrix} \begin{pmatrix} e^j_L \\ T^-_L \end{pmatrix} \quad , \quad m^i_e \equiv \frac{v}{\sqrt{2}} y^i_E \tag{5.31}$$

We choose y_T to be real so that this mass matrix is real. We can diagonalized it by two orthogonal matrices

$$\begin{pmatrix} e^i_L \\ T^-_L \end{pmatrix} \to O_- \begin{pmatrix} e^i_L \\ T^-_L \end{pmatrix} \tag{5.32}$$

In the same approximation as before

$$O_- = \begin{pmatrix} \mathbf{1} & \epsilon^- \\ -\epsilon^{-T} & 1 \end{pmatrix} \quad , \quad \epsilon^-_i = \frac{m_T v y^i_T}{m_T^2 - (m^i_e)^2} \tag{5.33}$$

and

$$\begin{pmatrix} \bar{e}_R & \bar{T}^-_R \end{pmatrix} \to \begin{pmatrix} \bar{e}_R & \bar{T}^-_R \end{pmatrix} O_+^T \tag{5.34}$$

with

$$O_+ = \begin{pmatrix} \mathbf{1} & \epsilon^+ \\ -\epsilon^{+T} & 1 \end{pmatrix} \quad , \quad \epsilon_i^+ = \frac{m_E^i v y_T^i}{m_T^2 - (m_e^i)^2} \tag{5.35}$$

5.3.5 Interaction in the mass eigenbasis

We will eventually interested in the decay rates of the triplets into light lepton and a gauge boson. These go through one power of the small Dirac Yukawa couplings y_T^i. It is thus enough at leading order to make the following replacements in Eq. (5.25), coming from Eq. (5.27)-(5.28).

$$\nu^j \rightarrow \nu^j + \epsilon_T^j T^3 + \epsilon_S^j S \tag{5.36}$$

$$T^3 \rightarrow T^3 - \epsilon_T^j \nu^j \tag{5.37}$$

$$S \rightarrow S - \epsilon_S^j \nu^j \tag{5.38}$$

and from Eq. (5.32)-(5.35)

$$e_L^j \rightarrow e_L^j + \sqrt{2}\epsilon_T^j T_L^- \tag{5.39}$$

$$e_R^j \rightarrow e_R^j + \sqrt{2}\frac{m_e^j}{m_T}\epsilon_T^j T_R^- \tag{5.40}$$

$$T_L^- \rightarrow T_L^- - \sqrt{2}\epsilon_T^j e_L^j \tag{5.41}$$

$$T_R^- \rightarrow T_R^- - \sqrt{2}\frac{m_e^j}{m_T}\epsilon_T^j e_R^j \tag{5.42}$$

After substitution we will have the following kinetic terms in the Lagrangian at $\mathcal{O}((\frac{y_X v}{m_X})^2)$

$$\begin{aligned} \mathcal{L}_{kin} &= i\overline{\nu}_\alpha \partial\!\!\!/ (\delta^{\alpha\beta} + \epsilon_T^\alpha \epsilon_T^\beta + \epsilon_S^{*\alpha}\epsilon_S^\beta)\nu_\beta + i\bar{e}_{\alpha L}\partial\!\!\!/ (\delta^{\alpha\beta} + 2\epsilon_T^\alpha\epsilon_T^\beta)e_{\beta L} + \bar{e}_R^\alpha \partial\!\!\!/ e_R^\alpha \\ &+ i\bar{T}_L^- \partial\!\!\!/ (1 + 2\epsilon_T^\alpha\epsilon_T^\alpha)T_L^- + i\bar{T}_R^- \partial\!\!\!/ T_R^- + i\bar{T}^3 \partial\!\!\!/ (1 + \epsilon_T^\alpha\epsilon_T^\alpha)T^3 + i\bar{S}\partial\!\!\!/ (1 + \epsilon_S^{*\alpha}\epsilon_S^\alpha)S \end{aligned} \tag{5.43}$$

Now in order to acquire canonically normalized kinetic terms we need to normalize the neutrino and charged lepton fieds. We make the following redefinitions on the fields

$$\begin{aligned}
\nu_\alpha &\rightarrow \nu'_\alpha \equiv \left(\delta^{\alpha\beta} + \frac{1}{2}\epsilon_T^\alpha \epsilon_T^\beta + \frac{1}{2}\epsilon_S^{*\alpha}\epsilon_S^\beta\right)\nu_\beta && (5.44)\\
S &\rightarrow S' \equiv \left(1 + \frac{1}{2}\epsilon_S^{*\alpha}\epsilon_S^\alpha\right) S && (5.45)\\
T^3 &\rightarrow T'^3 \equiv (1 + \frac{1}{2}\epsilon_T^\alpha \epsilon_T^\alpha) T^3 && (5.46)\\
e_{\alpha L} &\rightarrow e'_{\alpha L} \equiv \left(\delta^{\alpha\beta} + \epsilon_T^\alpha \epsilon_T^\beta\right) e_{\beta L} && (5.47)\\
T_L^- &\rightarrow T_L'^- \equiv (1 + \epsilon_T^\alpha \epsilon_T^\alpha)\, T_L^- && (5.48)
\end{aligned}$$

After this normalisation it is clear that there will be no mixing on electromagnetic interaction since it is proportional to kinetic terms, the mixing will only appear in the interaction with weak forces. Now we can write all the weak interactions after mixing and normalisation, we will have

$$\mathcal{L}_{kin} = \mathcal{L}_{CC} + \mathcal{L}_{NC} + \mathcal{L}_h + \mathcal{L}_\eta + \mathcal{L}_{\phi^-} \quad (5.49)$$

where

$$\mathcal{L}_{CC} = -\frac{g}{\sqrt{2}} \begin{pmatrix} \bar{e}_i & \bar{T}^- & \bar{S} \end{pmatrix} \gamma^\mu W_\mu^- \left(g_L^{CC} \gamma_L + \sqrt{2} g_R^{CC} \gamma_R \right) \begin{pmatrix} \nu_j \\ T^3 \\ S \end{pmatrix} + h.c. \tag{5.50}$$

$$\mathcal{L}_{NC} = -\frac{g}{c_W} \begin{pmatrix} \bar{e}_i & \bar{T}^- & \bar{S} & \bar{\nu}^i & \bar{T}^3 \end{pmatrix} \gamma^\mu Z_\mu \left(g_L^{NC} \gamma_L + g_R^{NC} \gamma_R \right) \begin{pmatrix} e_j \\ T^- \\ S \\ \nu^j \\ T^3 \end{pmatrix} \tag{5.51}$$

$$\begin{aligned} \mathcal{L}_h &= \frac{g}{2m_W} h \begin{pmatrix} \bar{e}_i & \bar{T}^- \end{pmatrix} \left(g_L^{hC} \gamma_L + g_R^{hC} \gamma_R \right) \begin{pmatrix} e_j \\ T^- \end{pmatrix} \\ &+ \frac{g}{2m_W} h \begin{pmatrix} \bar{S} & \bar{\nu}^i & \bar{T}^3 \end{pmatrix} \left(g_L^{hN} \gamma_L + g_R^{hN} \gamma_R \right) \begin{pmatrix} S \\ \nu^i \\ \bar{T}^3 \end{pmatrix} \end{aligned} \tag{5.52}$$

$$\begin{aligned} \mathcal{L}_\eta &= i\eta \frac{g}{2m_W} \begin{pmatrix} \bar{e}_i & \bar{T}^- \end{pmatrix} \left(g_L^{\eta C} \gamma_L + g_R^{\eta C} \gamma_R \right) \begin{pmatrix} e_j \\ T^- \end{pmatrix} \\ &+ i\eta \frac{g}{2m_W} \begin{pmatrix} \bar{S} & \bar{\nu}^i & \bar{T}^3 \end{pmatrix} \left(g_L^{\eta N} \gamma_L + g_R^{\eta N} \gamma_R \right) \begin{pmatrix} S \\ \nu^i \\ T^3 \end{pmatrix} \end{aligned} \tag{5.53}$$

$$\begin{aligned} \mathcal{L}_{\phi^-} = & -\frac{g}{\sqrt{2} m_W} \phi^- \bar{e}^i \Big[(g_{L\nu}^{\phi^-} \gamma_L + g_{R\nu}^{\phi^-} \gamma_R) \nu + (g_{LT}^{\phi -} \gamma_L + g_{RT}^{\phi -} \gamma_R) T^3 \\ & + (g_{LS}^{\phi -} \gamma_L + g_{RS}^{\phi -} \gamma_R) S \Big] + h.c. \end{aligned} \tag{5.54}$$

with $\gamma_{L,R} = \frac{1\pm\gamma_5}{2}$ and

$$g_L^{CC} = \begin{pmatrix} (\delta^{ij} + \frac{1}{2}\epsilon_T^i \epsilon_T^j) & -\epsilon_T^i & \epsilon_S^i \\ 0 & \sqrt{2} & \sqrt{2}\epsilon_T^j \epsilon_S^j \\ 0 & 0 & 0 \end{pmatrix} \tag{5.55}$$

$$g_R^{CC} = \begin{pmatrix} 0 & -\frac{m_e^i}{m_T}\epsilon_T^i & 0 \\ -\frac{1}{\sqrt{2}}\epsilon_T^i & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.56}$$

$$g_L^{NC} = \begin{pmatrix} (s_W^2 - \frac{1}{2})\delta^{ij} - \epsilon_T^i\epsilon_T^j & \frac{1}{\sqrt{2}}\epsilon_T^i & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}}\epsilon_T^i & -c_W^2 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}\epsilon_S^{*i}\epsilon_S^i & \frac{1}{2}\epsilon_S^{*i} & \frac{1}{2}\epsilon_S^{*i}\epsilon_T^i \\ 0 & 0 & \frac{1}{2}\epsilon_S^i & \frac{1}{2}(\delta^{ij} - \epsilon_T^i\epsilon_T^j - \epsilon_S^{*i}\epsilon_S^j) & \frac{1}{2}\epsilon_T^i \\ 0 & 0 & \frac{1}{2}\epsilon_T^i\epsilon_S^i & \frac{1}{2}\epsilon_T^i & \frac{1}{2}\epsilon_T^i\epsilon_T^i \end{pmatrix} \tag{5.57}$$

$$g_R^{NC} = \begin{pmatrix} s_W^2 & \sqrt{2}\frac{m_e^i}{m_T}\epsilon_T^i & 0 & 0 & 0 \\ \sqrt{2}\frac{m_e^i}{m_T}\epsilon_T^i & -c_W^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.58}$$

$$g_L^{hC} = \begin{pmatrix} -m_e^j(\delta^{ij} - 3\epsilon_T^i\epsilon_T^j) & -m_e^i\frac{y_T^i v}{m_T} \\ -\frac{m_e^{i2}}{m_T^2}y_T^i v - y_T^j v\left(\delta^{ij} - \epsilon_T^i\epsilon_T^j\right) & -\frac{m_e^{i2}}{m_T}\epsilon_T^i\epsilon_T^j - 2\frac{y_T^i v}{\sqrt{2}}\epsilon_T^i \end{pmatrix} \tag{5.59}$$

$$g_L^{hN} = \begin{pmatrix} -\frac{y_S^i v}{\sqrt{2}}\epsilon_S^{*i} & -\frac{y_S^{*j} v}{\sqrt{2}}\left[\delta^{ij}\left(1 + \frac{\epsilon_T^{*k}\epsilon_S^k}{2}\right) + \frac{\epsilon_T^i\epsilon_T^j}{2} + \frac{\epsilon_S^{*i}\epsilon_S^j}{2}\right] & -\frac{y_S^{*i} v}{\sqrt{2}}\epsilon_T^i \\ \frac{y_T^i v}{\sqrt{2}}\epsilon_S^{*i}\epsilon_T^i + \frac{y_S^{*i} v}{\sqrt{2}}\epsilon_S^{*i}\epsilon_S^i & \frac{y_T^i v}{\sqrt{2}}\epsilon_T^i + \frac{y_S^i v}{\sqrt{2}}\epsilon_S^i & \frac{y_S^{*i} v}{\sqrt{2}}\epsilon_S^{*i}\epsilon_T^i + \frac{y_T^i v}{\sqrt{2}}\epsilon_T^i\epsilon_T^i \\ -\frac{y_T^i v}{\sqrt{2}}\epsilon_S^i & -\frac{y^j v}{\sqrt{2}}\left[\delta^{ij}\left(1 + \frac{\epsilon_T^k\epsilon_T^k}{2}\right) + \frac{\epsilon_T^i\epsilon_T^j}{2} + \frac{\epsilon_T^{*i}\epsilon_S^j}{2}\right] & -\frac{y_T^i v}{\sqrt{2}}\epsilon_T^i \end{pmatrix} \tag{5.60}$$

$$g_R^{hC} = \begin{pmatrix} -m_e^j(\delta^{ij} - 3\epsilon_T^i\epsilon_T^j) & -y_T^j v(\delta^{ij} - \epsilon_T^i\epsilon_T^j) - \frac{m_e^{i2}}{m_T^2}y_T^i v \\ -m_e^i\frac{y_T^i v}{m_T} & -\frac{m_e^{i2}}{m_T}\epsilon_T^i\epsilon_T^i - \frac{2y_T^i v}{\sqrt{2}}\epsilon_T^i \end{pmatrix} \tag{5.61}$$

$$g_R^{hN} = \begin{pmatrix} -\frac{y_S^{*i} v}{\sqrt{2}}\epsilon_S^i & \frac{y_T^i v}{\sqrt{2}}\epsilon_S^{*i}\epsilon_S^i + \frac{y_S^i v}{\sqrt{2}}\epsilon_S^{*i}\epsilon_S^i & -\frac{y_T^i v}{\sqrt{2}}\epsilon_S^{*i} \\ -\frac{y_S^j v}{\sqrt{2}}\left[\delta^{ij}\left(1 + \frac{\epsilon_S^{*k}\epsilon_S^k}{2}\right) + \frac{\epsilon_T^i\epsilon_T^j}{2} + \frac{\epsilon_S^{*i}\epsilon_S^j}{2}\right] & \frac{y_T^i v}{\sqrt{2}}\epsilon_S^i + \frac{y_S^{*i} v}{\sqrt{2}}\epsilon_T^i + \frac{y_S^{*i} v}{\sqrt{2}}\epsilon_S^{*i} & -\frac{y_T^i v}{\sqrt{2}}\left[\delta^{ij}\left(1 + \frac{\epsilon_T^k\epsilon_T^k}{2}\right) + \frac{\epsilon_T^i\epsilon_T^j}{2} + \frac{\epsilon_S^{*i}\epsilon_S^j}{2}\right] \\ -\frac{y_S^i v}{\sqrt{2}}\epsilon_T^i & \frac{y_T^i v}{\sqrt{2}}\epsilon_T^i\epsilon_T^i - \frac{y_S^i v}{\sqrt{2}}\epsilon_T^i\epsilon_S^i & \frac{y_T^i v}{\sqrt{2}}\epsilon_T^i \end{pmatrix} \tag{5.62}$$

$$g_L^{\eta C} = \begin{pmatrix} m_e^i(\delta^i j + \epsilon_T^i\epsilon_T^j) & m_e^i\frac{y_T^i v}{m_T} \\ \frac{m_e^{i2}}{m_T^2}y_T^i v - y_T^i v\left(\delta^{ij} + \epsilon_T^i\epsilon_T^j\right) & \frac{2m_e^{i2}}{m_T}\epsilon_T^i\epsilon_T^i - \frac{2y_T^i v}{\sqrt{2}}\epsilon_T^i \end{pmatrix} \tag{5.63}$$

$$g_L^{\eta N} = \begin{pmatrix} \frac{y_S^i v}{\sqrt{2}}\epsilon_S^{*i} & -\frac{y_S^{*j} v}{\sqrt{2}}\left[\delta^{ij}\left(1 + \frac{\epsilon_T^{*k}\epsilon_S^k}{2}\right) + \frac{\epsilon_T^i\epsilon_T^j}{2} + \frac{\epsilon_S^{*i}\epsilon_S^j}{2}\right] & -\frac{y_S^{*i} v}{\sqrt{2}}\epsilon_T^i \\ \frac{y_T^i v}{\sqrt{2}}\epsilon_S^i\epsilon_T^i + \frac{y_S^{*i} v}{\sqrt{2}}\epsilon_S^{*i}\epsilon_S^i & -\frac{y_T^i v}{\sqrt{2}}\epsilon_T^i - \frac{y_S^i v}{\sqrt{2}}\epsilon_S^i & \frac{y_T^i v}{\sqrt{2}}\epsilon_T^i\epsilon_T^i + \frac{y_S^{*i} v}{\sqrt{2}}\epsilon_T^i\epsilon_S^{*i} \\ -\frac{y_T^i v}{\sqrt{2}}\epsilon_S^i & -\frac{y^i v}{\sqrt{2}}\left[\delta^{ij}\left(1 + \frac{\epsilon_T^k\epsilon_T^k}{2}\right) + \frac{\epsilon_T^i\epsilon_T^j}{2} + \frac{\epsilon_S^{*i}\epsilon_S^j}{2}\right] & -\frac{y_T^i v}{\sqrt{2}}\epsilon_T^i \end{pmatrix} \tag{5.64}$$

$$g_R^{\eta C} = \begin{pmatrix} -m_e^i\left(\delta^{ij}+\epsilon_T^i\epsilon_T^j\right) & -\frac{m_e^{i2}}{m_T^2}y_T^i v + y_T^i v\left(\delta^{ij}+\epsilon_T^i\epsilon_T^j\right) \\ -m_e^i\frac{y_T^i v}{m_T} & -\frac{2m_e^{i2}}{m_T}\epsilon_T^i\epsilon_T^i + \frac{2y_T^i v}{\sqrt{2}}\epsilon - T^i \end{pmatrix} \tag{5.65}$$

$$g_R^{\eta N} = \left(\begin{matrix} -\frac{y_S^{*i}v}{\sqrt{2}}\epsilon_S^i & -\frac{y_T^i v}{\sqrt{2}}\epsilon_S^i\epsilon_T^i - \frac{y_S^i v}{\sqrt{2}}\epsilon_S^{*i}\epsilon_S^i \\ \frac{y_S^j v}{\sqrt{2}}\left[\delta^{ij}\left(1+\frac{\epsilon_T^k\epsilon_S^{*k}}{2}\right)+\frac{\epsilon_T^i\epsilon_T^j}{2}+\frac{\epsilon_S^i\epsilon_S^{*j}}{2}\right] & \frac{y_T^i v}{\sqrt{2}}\epsilon_T^i + \frac{y_S^{*i}v}{\sqrt{2}}\epsilon_S^{*i} \\ \frac{y_S^i v}{\sqrt{2}}\epsilon_T^i & -\frac{y_T^i v}{\sqrt{2}}\epsilon_T^i\epsilon_T^i - \frac{y_S^i v}{\sqrt{2}}\epsilon_T^i\epsilon_S^i \end{matrix} \right. \cdots$$

$$\cdots \left. \begin{matrix} \frac{y_T^i v}{\sqrt{2}}\epsilon_S^{*i} \\ \frac{y^i v}{\sqrt{2}}\left[\delta^{ij}\left(1+\frac{\epsilon_T^k\epsilon_T^k}{2}\right)+\frac{\epsilon_T^i\epsilon_T^j}{2}+\frac{\epsilon_S^i\epsilon_S^{*j}}{2}\right] \\ \frac{y_T^i v}{\sqrt{2}}\epsilon_T^i \end{matrix} \right) \tag{5.66}$$

$$g_{L\nu}^{\phi^-} = m_e^i(1-\frac{1}{2}\epsilon_T^i\epsilon_T^j-\frac{1}{2}\epsilon_S^{*i}\epsilon_S^j) \tag{5.67}$$

$$g_{R\nu}^{\phi^-} = (\delta^{i\alpha}-\epsilon_T^i\epsilon_T^\alpha)m_\nu^{\alpha\beta}(\delta^{\beta j}-\frac{1}{2}\epsilon_T^\beta\epsilon_T^j-\frac{1}{2}\epsilon_S^{*\beta}\epsilon_S^j) \tag{5.68}$$

$$g_{LT}^{\phi^-} = m_e^i\epsilon_T^i \tag{5.69}$$

$$g_{RT}^{\phi^-} = (1-\epsilon_T^i\epsilon_T^j)y_T^j\frac{v}{\sqrt{2}}(1-\frac{1}{2}\epsilon_T^j\epsilon_T^j)+2m_T\epsilon_T^i\epsilon_T^j\epsilon_T^j \tag{5.70}$$

$$g_{LS}^{\phi^-} = m_e^i\epsilon_S^{*i} \tag{5.71}$$

$$g_{RS}^{\phi^-} = (1-\epsilon_T^i\epsilon_T^j)y_T^i\frac{v}{\sqrt{2}}(1-\frac{1}{2}\epsilon_S^{*i}\epsilon_S^j)+2m_T\epsilon_T^i\epsilon_T^j\epsilon_S^{*j} \tag{5.72}$$

5.4 $T^+ - T^3$ mass difference

For a nonvanishing and positive mass split $\Delta m_T \equiv m_{T^+} - m_{T^3}$ the charged triplet fermion can decay into a neutral one and an (off-shell) W.

One gets for Δm_T at the one-loop level

$$\Delta m_T = \frac{\alpha_2}{2\pi}\frac{m_W^2}{m_T}\left[f\left(\frac{m_T^2}{m_Z^2}\right)-f\left(\frac{m_T^2}{m_W^2}\right)\right] \tag{5.73}$$

where

$$f(y) = \frac{1}{4y}\ln y - \left(1+\frac{1}{2y}\right)\sqrt{4y-1}\arctan\sqrt{4y-1} \tag{5.74}$$

We will get $\Delta m_T \approx 132\,\text{MeV}$ when $m_T = m_Z$ and it is $\Delta m_T \approx 146\,\text{MeV}$ when $m_T \to \infty$. The value of Δm_T for varying m_T is shown on the Fig. 5.1.

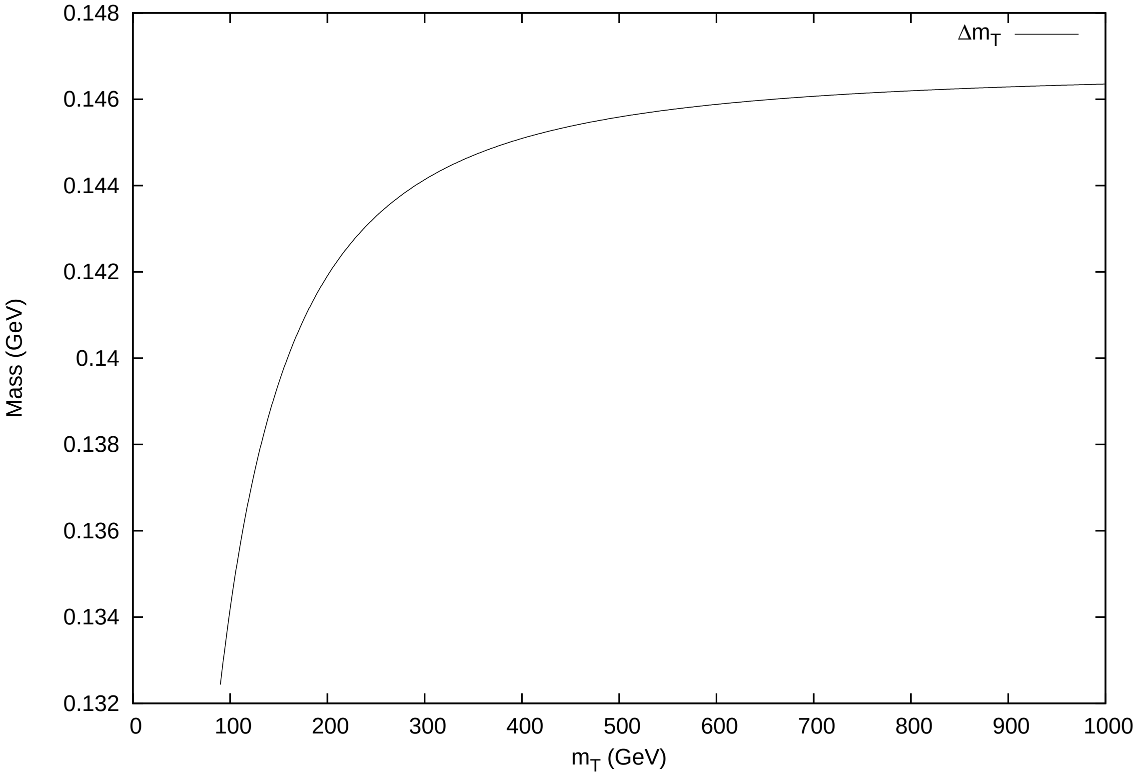


Figure 5.1: Δm_T on various m_T

Chapter 6

Phenomenology

In this chapter we study some of the new phenomenology of the light fermion triplet. In addition, the possibility of light triplet have made some of this process large enough to be seen on LHC. Refer to the appendices for more detailed derivation of the results given here.

6.1 Triplet decay

Triplet fermions can decay to gauge boson and lepton with the following decay rate

$$\Gamma(T^- \to Ze^k) = \frac{1}{32\pi}|y^k|^2 \frac{(m_T^2 - m_Z^2)^2(m_T^2 + 2m_Z^2)}{m_T^5} \tag{6.1}$$

$$\Gamma(T^- \to W^-\nu^k) = \frac{1}{16\pi}|y^k|^2 \frac{(m_T^2 - m_W^2)^2(m_T^2 + 2m_W^2)}{m_T^5} \tag{6.2}$$

$$\begin{aligned}\Gamma(T^3 \to W^+e^k) &= \Gamma(T^3 \to W^-\bar{e}^k) = \\ &= \frac{1}{32\pi}|y^k|^2 \frac{(m_T^2 - m_W^2)^2(m_T^2 + 2m_W^2)}{m_T^5}\end{aligned} \tag{6.3}$$

$$\Gamma(T^3 \to Z\nu_k) = \frac{1}{32\pi}|y_T^k|^2 \frac{(m_T^2 - m_Z^2)^2(m_T^2 + 2m_Z^2)}{m_T^5} \tag{6.4}$$

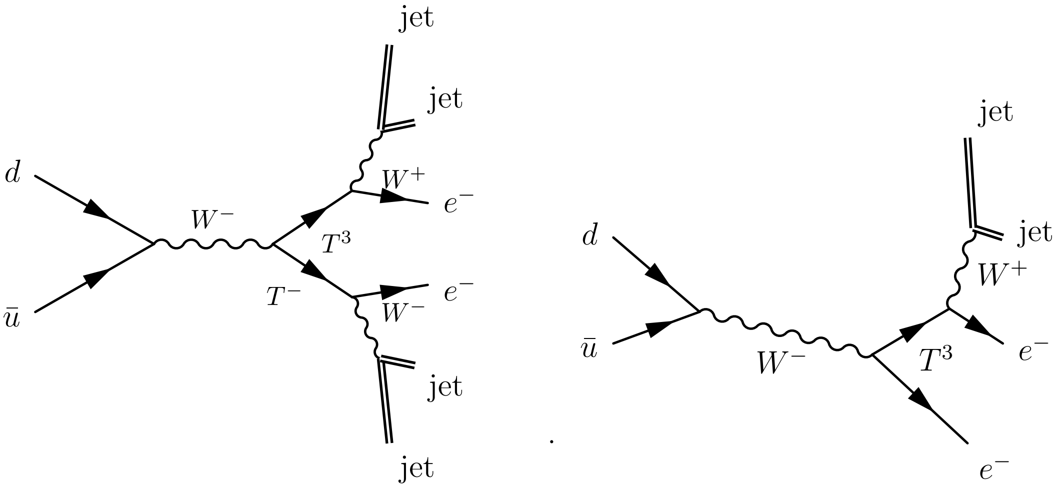


Figure 6.1: Diagrams giving lepton number violating process

The triplet may also decay into higgs particle if $m_T > m_H$ with the following decay rate

$$\Gamma(T \to he) = \Gamma(T \to h\nu) = \frac{1}{32\pi}|y_T^j|^2 \frac{(m_T^2 - m_H^2)^2}{m_T^3} \tag{6.5}$$

In experiment such as LHC one might be able to produce triplet fermions whose decays will give interesting signatures of dilepton with same signs accompanied by 2 or 4 jets coming from diagrams shown in fig. 6.1.

We can obtain more information about this decay from neutrino oscillation data. From the known formula [13] (see Appendix) we have for hierarchical case ($m_1^\nu = 0$)

$$\frac{vy_T^{i*}}{\sqrt{2}} = i\sqrt{m_T}\left(U_{i2}\sqrt{m_2^\nu}\cos z \pm U_{i3}\sqrt{m_3^\nu}\sin z\right) \tag{6.6}$$

$$\frac{vy_S^{i*}}{\sqrt{2}} = -i\sqrt{m_S}\left(U_{i2}\sqrt{m_2^\nu}\sin z \mp U_{i3}\sqrt{m_3^\nu}\cos z\right) \tag{6.7}$$

or in the case of inverse hierarchy ($m_3^\nu = 0$)

$$\frac{vy_T^{i*}}{\sqrt{2}} = i\sqrt{m_T}\left(U_{i1}\sqrt{m_1^\nu}\cos z \pm U_{i2}\sqrt{m_2^\nu}\sin z\right) \tag{6.8}$$

$$\frac{vy_S^{i*}}{\sqrt{2}} = -i\sqrt{m_S}\left(U_{i1}\sqrt{m_1^\nu}\sin z \mp U_{i2}\sqrt{m_2^\nu}\cos z\right) \tag{6.9}$$

where z is a complex number and U the lepton mixing matrix that diagonalize the neutrino mass matrix.

$$m^\nu = U^* \begin{pmatrix} m_1^\nu & 0 & 0 \\ 0 & m_2^\nu & 0 \\ 0 & 0 & m_3^\nu \end{pmatrix} U^\dagger \tag{6.10}$$

If we suppose the third neutrino is massless then Eq. (6.6) -(6.9) will give the following relation (in the limit $\theta_{13}^{PMNS} = 0$)

$$\text{normal hierarchy} \quad : \quad (y_T^\mu \cos\theta_{23} - y_T^\tau \sin\theta_{23}) = \frac{y_T^e}{\tan\theta_{12}} \tag{6.11}$$

$$\text{inverse hierarcy} \quad : \quad y_T^\tau = -\tan\theta_{23} y_T^\mu \tag{6.12}$$

Thus for case of inverse hierarchy we should have

$$\Gamma(T \to \tau + x) = \tan^2\theta_{23}\Gamma(T \to \mu + x) \tag{6.13}$$

6.2 $e^j \to 3e$ decay

Lepton with different flavour can mix while interacting with Z boson. The interaction comes from

$$\delta\mathcal{L} = -\frac{eZ_\mu}{2s_W c_W}\left[\bar{e}_L^j \gamma^\mu((2s_W^2 - 1)\delta^{jk} - 2\epsilon_T^j \epsilon_T^k)e_L^k + 2s_W^2 \bar{e}_R^j \gamma^\mu e_R^j\right] Z_\mu \tag{6.14}$$

This may give rise to $e^j \to$ and $e^j \to \bar{e}^k e^l e^l$ decay mode with

$$\Gamma(e^j \to \bar{e}^k e^k e^l) = \frac{m_j^5}{24(8\pi)^3 m_T^4}(y_T^j y_T^k)^2(g_V^2 + 1) \tag{6.15}$$

$$\Gamma(e^j \to \bar{e}^l e^l e^l) = \frac{m_j^5}{48(8\pi)^3 m_T^4}(y_T^j y_T^k)^2\left[4(g_V - 1)^2 + 2(g_V + 1)^2\right] \tag{6.16}$$

$$\Gamma(e^j \to \bar{e}^k e^l e^l) = \frac{m_j^5 v^4}{3(8\pi)^3}\left(\frac{y_T^j y_T^k y_T^l y_T^l}{m_T^4}\right)^2 \tag{6.17}$$

where $g_V = 4s_W^2 - 1$.

6.2.1 μ decay

The main decay mode of μ is $\mu \to \nu_\mu \bar{\nu}_e e$ with branching ratio $\sim 100\%$. Its decay rate for that mode is given by

$$\Gamma(\mu \to \nu_\mu \bar{\nu}_e e) = \frac{m_\mu^5 g^4}{12(8\pi)^3 m_W^4} \sim \Gamma_{\text{tot}} \tag{6.18}$$

Then using Eq. (6.15) we can estimate the brancing ratio for $\mu \to \bar{e}ee$

$$BR(\mu \to \bar{e}ee) = \frac{\Gamma(\mu \to \bar{e}ee)}{\Gamma_{\text{tot}}} = \frac{m_W^4 (y_T^\mu y_T^e)^2}{4 m_T^4 g^4} \left[4(g_V - 1)^2 + 2(g_V + 1)^2\right] \tag{6.19}$$

This branching ratio has been measured on the experiment with[14]

$$BR(\mu \to \bar{e}ee)_{exp} < 1.0 \times 10^{-12} \tag{6.20}$$

Using this limit we can estimate the constraint of $y_T^\mu y_T^e$ as shown on fig. 6.2.

6.2.2 τ decay

Possible flavour violating decay mode for τ are $\tau \to \bar{\mu}\mu\mu$, $\tau \to \bar{\mu}\mu e$, $\tau \to \bar{\mu}ee$, $\tau \to \bar{e}\mu\mu$, $\tau \to \bar{e}\mu e$ and $\tau \to \bar{e}ee$.

Before calculating the decay rate for each τ flavour violating decay we need to find its total decay rate. τ goes into $e\bar{\nu}_e\nu_\tau$ with branching ratio 17.85%, therefore

$$\Gamma(\tau \to e\bar{\nu}_e\nu_\tau) = \frac{m_\tau^5 g^4}{12(8\pi)^3 m_W^4} = 0.18\Gamma_{\text{tot}} \tag{6.21}$$

using the same way as we did previously, we can then find the branching ratio which are

$$BR(\tau \to \bar{\mu}\mu\mu) = \frac{\Gamma(\tau \to \bar{\mu}\mu\mu)}{\Gamma_{\rm tot}} = 0.18\frac{m_W^4(y_T^\tau y_T^\mu)^2}{4m_T^4 g^4}\left[4(g_V+1)^2+2(g_V-1)^2\right] \tag{6.22}$$

$$BR(\tau \to \bar{e}e\mu) = \frac{\Gamma(\tau \to \bar{\mu}\mu e)}{\Gamma_{tot}} = 0.18\frac{m_W^4(y_T^\tau y_T^\mu)^2}{2m_T^4 g^4}(g_V^2+1) \tag{6.23}$$

$$BR(\tau \to \bar{e}ee) = \frac{\Gamma(\tau \to \bar{e}ee)}{\Gamma_{\rm tot}} = 0.18\frac{m_W^4(y_T^\tau y_T^e)^2}{4m_T^4 g^4}\left[4(g_V+1)^2+2(g_V-1)^2\right] \tag{6.24}$$

$$BR(\tau \to \bar{\mu}\mu e) = \frac{\Gamma(\tau \to \bar{e}ee)}{\Gamma_{tot}} = 0.18\frac{m_W^4(y_T^\tau y_T^e)^2}{4m_T^4 g^4}(g_V^2+1) \tag{6.25}$$

$$BR(\tau \to \bar{e}\mu\mu) = 0.18\frac{4m_W^4 v^4}{g^4}\left(\frac{y_T^\tau y_T^\mu y_T^e y_T^e}{m_T^4}\right)^2 \tag{6.26}$$

$$BR(\tau \to \bar{\mu}ee) = 0.18\frac{4m_W^4 v^4}{g^4}\left(\frac{y_T^\tau y_T^\mu y_T^e y_T^e}{m_T^4}\right)^2 \tag{6.27}$$

Thus we predicted the same branching ratio both for $\tau \to \bar{\mu}\mu\mu$ with $\tau \to \bar{e}e\mu$ and $\tau \to \bar{\mu}\mu e$ with $\tau \to \bar{e}ee$. Moreover given that $\epsilon_T^j \ll 1$, we should have $BR(\tau \to \bar{e}\mu\mu)$ and $BR(\tau \to \bar{\mu}ee)$ to be very small compared to other flavour violating mode.

The constraints for these branching ratio taken from experiments are given below[14]

Decay Mode	Branching Ratio
$\tau \to \bar{\mu}\mu\mu$	$< 3.2 \times 10^{-8}$
$\tau \to \bar{e}e\mu$	$< 2.7 \times 10^{-8}$
$\tau \to \bar{\mu}\mu e$	$< 3.7 \times 10^{-8}$
$\tau \to \bar{e}ee$	$< 3.6 \times 10^{-8}$
$\tau \to \bar{e}\mu\mu$	$< 2.3 \times 10^{-8}$
$\tau \to \bar{\mu}ee$	$< 2.0 \times 10^{-8}$

From this value we can estimate the constraints for y_T's as shown in fig. 6.3.

6.3 $e^i \to e^j\gamma$ decay

At one loop order, leptons may decay into other lepton of different flavour accompanied by gamma from the interaction with triplet and singlet fermion. The decay rate is given by

$$\Gamma(e^i \to e^j\gamma) = \frac{m_i^3}{16\pi}|A|^2 \tag{6.28}$$

where the coefficient A is given by

$$\begin{aligned} A &= \tfrac{m_1 g^2 e}{32\pi^2 m_W^2} \quad \left[\epsilon_T^2\epsilon_T^1\left(\frac{-13}{12} - 1,64 + \mathcal{A}(x_T) + \mathcal{B}(y_T) + \mathcal{C}(z_T)\right)\right. \\ &\quad \left. + \epsilon_S^{*2}\epsilon_S^1\left(\frac{5}{12} + \mathcal{D}(x_S)\right)\right] \end{aligned} \tag{6.29}$$

where $x_T = \frac{m_T^2}{m_W^2}$, $y_T\frac{m_T^2}{m_Z^2} =$, $z_T = \frac{m_T^2}{m_h^2}$ and $x_S = \frac{m_S^2}{m_W^2}$ and

$$\mathcal{A}(x) = \frac{-39x^3 + 114x^2 - 93x - 6 - (24x^3 - 12x^2 - 6x)\ln(x)}{12(x-1)^4} \tag{6.30}$$

$$\mathcal{B}(x) = \frac{-30x^3 + 45x^2 + 18x - 33 - 18x(3x-4)\ln(x)}{12(x-1)^4} \tag{6.31}$$

$$\mathcal{C}(x) = \frac{8x^3 - 3x^2 - 12x + 7 - 6x(3x-1)\ln(x)}{12(x-1)^4} \tag{6.32}$$

$$\mathcal{D}(x) = \frac{69x^3 - 138x^2 + 85x - 16 - 42x^3\ln(x)}{12(x-1)^4} \tag{6.33}$$

In the calculation of the decay rate we have ignored the possibility of loop from quark and X, Y boson since their contribution will be very small. Moreover we can also take the singlet to be arbitrary heavy so that $m_S \gg m_T$ therefore the part proportional to ϵ_S can simply be ignored.

Once again we can evaluate the constraint on $y_T^i y_T^j$ from the branching ratio constraint taken from experiment. From experiment data we have[14]

Decay Mode	Branching Ratio
$\mu \to e\gamma$	1.2×10^{-11}
$\tau \to \mu\gamma$	6.8×10^{-8}
$\tau \to e\gamma$	1.1×10^{-7}

and $y_T^i y_T^j$ constraint are shown in Fig. 6.4,6.5 and 6.6 for $m_Z < m_T < 10^3 GeV$.

6.4 Comparison of $e^i \to e^j\gamma$ and $e^i \to 3e$ decay

As can be seen from Eq. (6.15) and Eq. (6.28) that $\Gamma(e^i \to \bar{e}^j e^j e^k)$ and $\Gamma(e^i \to e^j\gamma)$ are both proportional to $(y_T^i y_T^j)^2$ so that we can make comparison on them. Their relative branching ratio is given as following

$$\begin{aligned}
\frac{\Gamma(\mu \to e\gamma)}{\Gamma(\mu \to \bar{e}ee)} &= \frac{\Gamma(\tau \to e\gamma)}{\Gamma(\tau \to \bar{e}ee)} = \frac{\Gamma(\tau \to \mu\gamma)}{\Gamma(\tau \to \bar{\mu}\mu\mu)} \\
&= \frac{3e^2}{8\pi^2}\frac{|-\frac{13}{12} - 1,64 + \mathcal{A}(x_T) + \mathcal{B}(y_T) + \mathcal{C}(z_T)|^2}{[4(g_V - 1)^2 + 2(g_V + 1)^2]} \qquad (6.34)\\
\frac{\Gamma(\tau \to e\gamma)}{\Gamma(\tau \to \bar{\mu}\mu e)} &= \frac{\Gamma(\tau \to \mu\gamma)}{\Gamma(\tau \to \mu\bar{e}e)} \\
&= \frac{3e^2}{16\pi^2}\frac{|-\frac{13}{12} - 1,64 + \mathcal{A}(x_T) + \mathcal{B}(y_T) + \mathcal{C}(z_T)|^2}{g_V^2 + 1} \qquad (6.35)
\end{aligned}$$

Thus according to the $SU(5)$ model, we have predicted a smaller decay rate for $e^i \to e^j\gamma$ decay when compared to $e^i \to \bar{e}^j e^j e^k$ by order of at least 10^2.

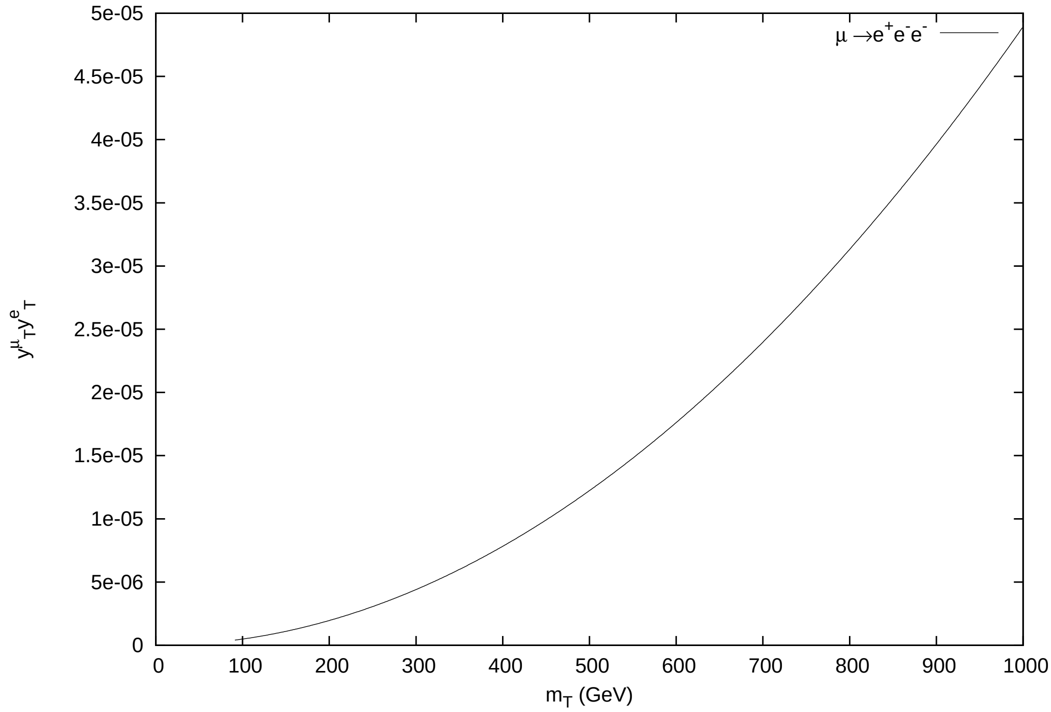


Figure 6.2: The constraint of $y_T^\mu y_T^e$ plotted as a function of m_T evaluated from $\mu \to \bar{e}ee$.

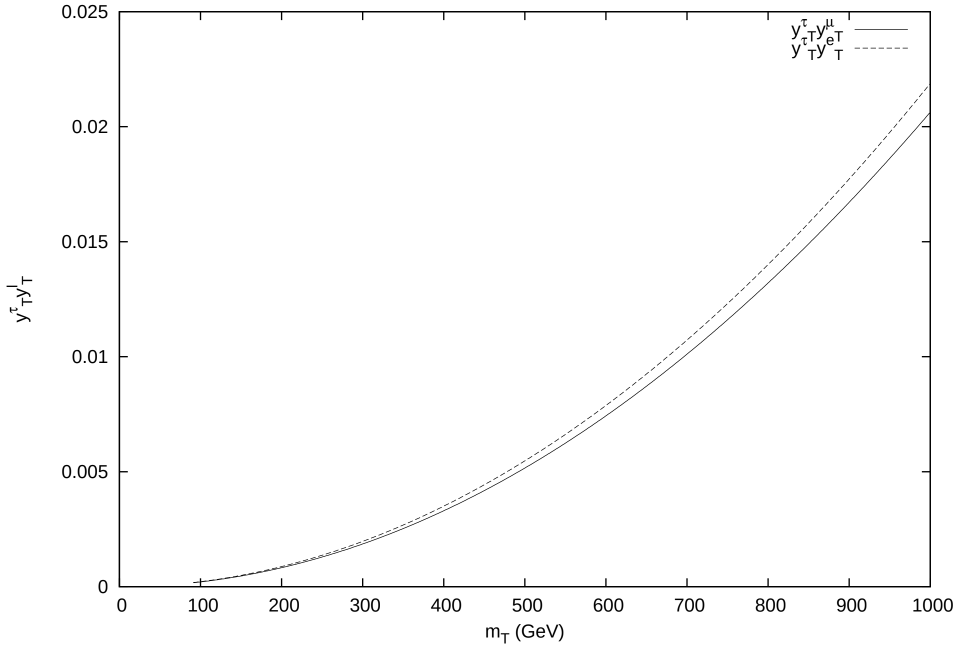


Figure 6.3: $y_T^\tau y_T^i$ constraint plotted as a function of m_T evaluated from $\tau \to \bar{\mu}\mu\mu$ and $\tau \to \bar{e}ee$.

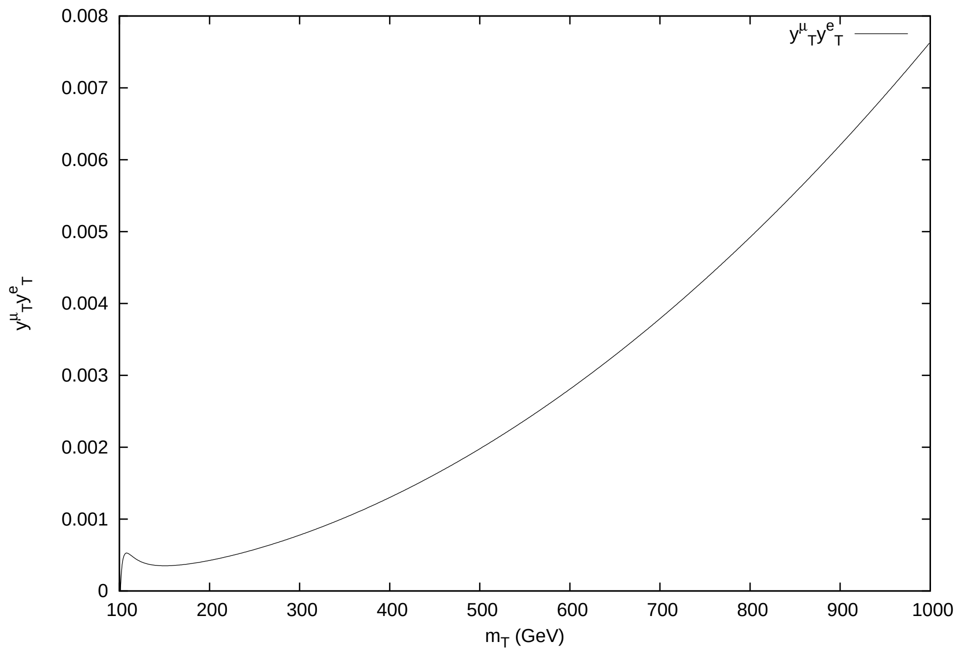


Figure 6.4: $y_T^\mu y_T^e$ constraint from $\mu \to e\gamma$ decay

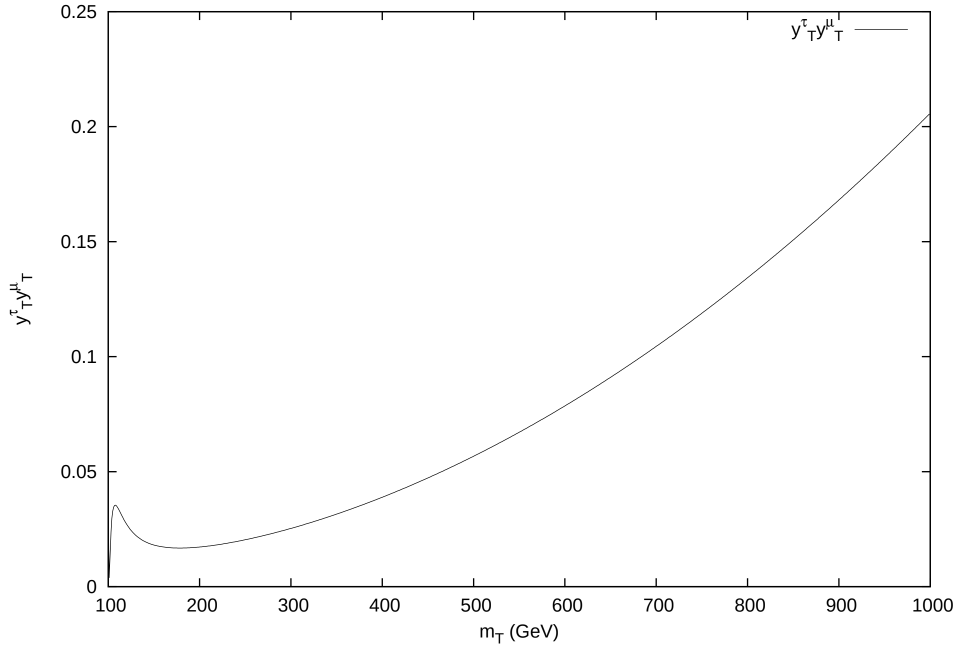


Figure 6.5: $y_T^\tau y_T^\mu$ constraint from $\tau \to \mu\gamma$ decay

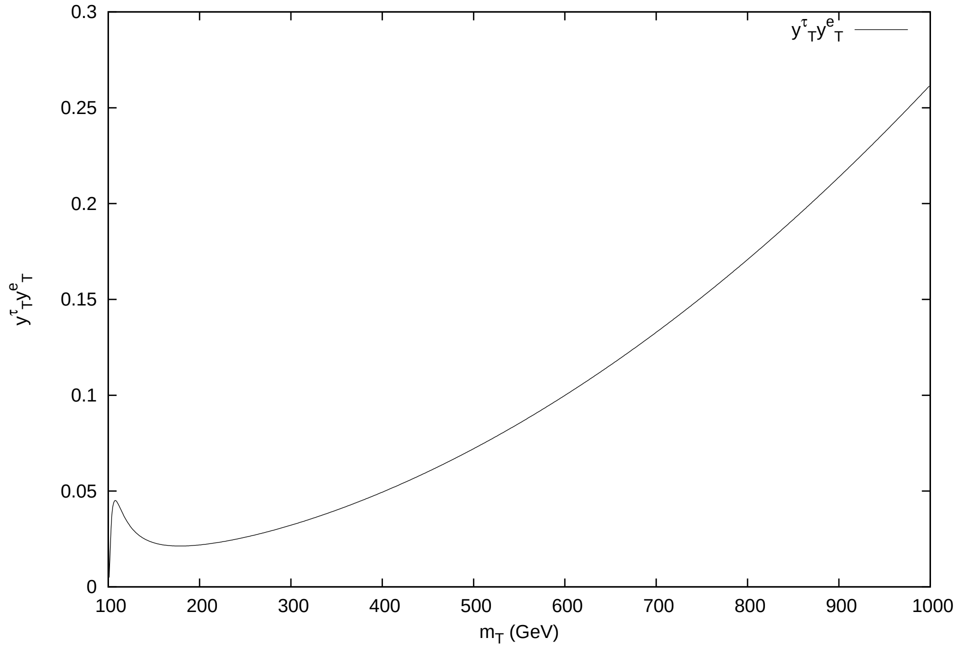


Figure 6.6: $y_T^\tau y_T^e$ constraint from $\tau \to e\gamma$ decay

Chapter 7

Concluding Remarks

The minimal $SU(5)$ Grand Unified Theory fails for two important reasons: the neutrinos are massless as in the minimal Standard Model, and the gauge coupling do not unify. As we discussed in this thesis, both problems are cured through the addition of an adjoint fermion representation. In turn this leads to an exciting prediction of a light $SU(2)$ triplet fermion with mass below TeV. The theory is thus directly testable in near future experiments.

The existence of triplet fermion can be checked from the appearance of signature in the experiment such as LHC by production of dilepton with same signs accompanied with 2 or 4 jets. We have calculated the total decay rate of triplets for every mode, however to make detailed evaluation on this process we will need further analysis using numerical simulation to make a direct comparison of the result with the experiment.

Another impotant consequence of a light triplet are lepton flavour violating processes. In the Standard Model augmented by non vanishing neutrino masses these effects are infinitesimally small due to the smalness of neutrino masses, so this can be an additional test of the theory. We have calculated the branching ratio two types of lepton flavour violation which may occurs from process such as $\mu \to e\gamma, \tau \to \mu\gamma, \tau \to e\gamma$ or $\mu \to \bar{e}ee, \tau \to \bar{\mu}\mu\mu, \tau \to \bar{e}ee$, etc. In addition, we have calculated the relative branching between $e^i \to e^k\gamma$ and $e^i \to \bar{e^k}e^le^l$ process with the result shown on Eq. (6.35), (6.36) and (6.37). With these results we have predicted the process of the type $e^i \to e^k\gamma$ should appear less frequently compared to $e^i \to \bar{e^k}e^le^l$.

Appendix A

$SU(5)$ Generator

The $SU(5)$ generator used in this work are defined as following.

$$\lambda_i = \begin{pmatrix} & & & 0 & 0 \\ & \lambda_i^{\{8\}} & & 0 & 0 \\ & & & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad i = 1, \ldots, 8$$

$$\lambda_9 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \lambda_{10} = \begin{pmatrix} 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_{11} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \lambda_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_{13} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \lambda_{14} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_{15} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad \lambda_{16} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_{17} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \lambda_{18} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 \\ 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_{19} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad \lambda_{20} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 \end{pmatrix}$$

$$\lambda_{20+i} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma_i & \\ 0 & 0 & 0 & & \end{pmatrix}, \qquad i = 1, \ldots, 3$$

$$\lambda_{24} = \frac{2}{\sqrt{15}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 & -\frac{3}{2} \end{pmatrix} \tag{A.1}$$

where $\lambda_i^{\{8\}}$ is the $SU(3)$ generator and σ_i is the Pauli matrix.

Appendix B

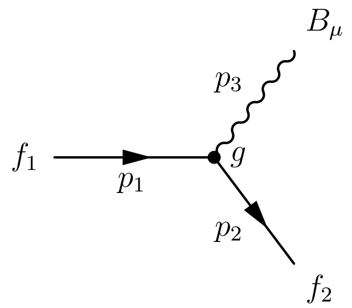
Tree Level Processes

B.1 $T \to f + B$

The triplets can decay into leptons and gauge bosons from the following interactions

$$
\begin{aligned}
\delta\mathcal{L}_{gauge} = \quad & \tfrac{-e}{2s_W c_W} Z_\mu \quad \left[\epsilon_T^j \bar{\nu}^j \gamma^\mu \gamma^5 T^3 + \sqrt{2}\epsilon_T^j (\bar{e}_L^j \gamma^\mu T_L^- + \overline{T^-}_L \gamma^\mu e_L^j)\right] \\
& + \tfrac{e}{s_W} W_\mu^- \quad \left[\frac{1}{\sqrt{2}} \epsilon_T^j \bar{e}_L^j \gamma^\mu T_L^3 + \epsilon_T^j \overline{T^-}_R \gamma^\mu \nu_R^j\right] \\
& + \tfrac{e}{s_W} W_\mu^+ \quad \left[\frac{1}{\sqrt{2}} \epsilon_T^j \overline{T^3}_L \gamma^\mu e_L^j + \epsilon_T^j \bar{\nu}_R^j \gamma^\mu T_R^-\right] \qquad \text{(B.1)}
\end{aligned}
$$

Most of these interaction has V-A current except for the interaction of T^3 and ν which is of the type of an axial vector current. We will first give the general decay rate for V-A interaction and axial vector interaction, then we plug the corresponding constant for each particular process we want to obtain.



For a general fermion f_1 decaying into a gauge boson B_μ and other fermion f_2 with V-A interaction, the amplitude is

$$\mathcal{M} = g\bar{u}_2^r(p_2)\gamma^\mu \frac{(1-\gamma^5)}{2} u_1^s(p_1)\epsilon_\mu^*(p_3) \tag{B.2}$$

so the amplitude square is

$$\begin{aligned} \overline{|\mathcal{M}|^2} &= \frac{g^2}{8} Tr\left[(\not{p}_2 - m_2)\gamma^\mu(1-\gamma^5)(\not{p}_1 - m_1)\gamma^\nu(1-\gamma^5)\right]\left(-g_{\mu\nu} + \frac{p_{3\mu}p_{3\nu}}{m_B^2}\right) \\ &= \frac{g^2}{4} Tr\left[\not{p}_2\gamma^\mu\not{p}_1\gamma^\nu(1-\gamma^5)\right]\left(-g_{\mu\nu} + \frac{p_{3\mu}p_{3\nu}}{m_B^2}\right) \end{aligned} \tag{B.3}$$

The trace on Eq. (B.3) can be evaluated and we obtain

$$Tr[\not{p}_2\gamma^\mu\not{p}_1\gamma^\nu] = 4(p_1^\mu p_2^\nu + p_2^\mu p_1^\nu - (p_1\cdot p_2)g^{\mu\nu}) \tag{B.4}$$

$$Tr[\not{p}_2\gamma^\mu\not{p}_1\gamma^\nu\gamma^5] = -4ip_{2\alpha}p_{1\beta}\epsilon^{\alpha\mu\beta\nu} \tag{B.5}$$

Therefore Eq. (B.3) becomes

$$\begin{aligned} \overline{|\mathcal{M}|^2} &= 2g^2\left[2(p_1\cdot p_2) + \frac{2(p_1\cdot p_3)(p_2\cdot p_3) - (p_1\cdot p_2)m_B^2}{m_B^2}\right] \\ &= 2g^2\left[\frac{(p_1\cdot p_2)m_B^2 + 2(p_1\cdot p_3)(p_2\cdot p_3)}{m_B^2}\right] \end{aligned} \tag{B.6}$$

If we work in the frame where the initial particle is at rest and using the approximation where $m_2^2 \sim 0$ we will have

$$p_1 = p_2 + p_3 \tag{B.7}$$

$$2(p_1\cdot p_3) = -(p_1-p_3)^2 + p_1^2 + p_3^2 = m_1^2 + m_B^2 \tag{B.8}$$

$$2(p_1\cdot p_2) = -(p_1-p_2)^2 + p_1^2 + p_2^2 = m_1^2 - m_B^2 \tag{B.9}$$

$$2(p_2\cdot p_3) = (p_2+p_3)^2 - p_2^2 - p_3^2 = m_1^2 - m_B^2 \tag{B.10}$$

So Eq. (B.6) becomes

$$\overline{|\mathcal{M}|^2} = g^2\left[\frac{(m_1^2 + 2m_B^2)(m_1^2 - m_B^2)}{m_B^2}\right] \tag{B.11}$$

We can now find the decay rate by plugging the amplitude from the generic formula for 2 body decay

$$\Gamma = \frac{1}{16\pi}\overline{|\mathcal{M}|^2}\frac{|\mathbf{p}_2|}{m_1^2} \tag{B.12}$$

where $|\mathbf{p}_2|$ is the 3-momentum of f_2 which is found by solving energy conservation relation to be equal to

$$|\mathbf{p}_2| = \frac{m_1^2 - m_B^2}{2m_1} \tag{B.13}$$

finally after plugging the amplitude and Eq. (B.13) to Eq. (B.12) the decay rate is given by

$$\Gamma = \frac{1}{32\pi}\left(\frac{g}{m_B}\right)^2 \frac{(m_1^2 - m_B^2)^2(m_1^2 + 2m_B^2)}{m_1^3} \tag{B.14}$$

We can simply use this formula and put the appropriate coupling constant for each process to obtain its decay rate. We will obtain the following result

$$\begin{aligned}
\Gamma(T^- \to Ze^k) &= \frac{1}{32\pi}|y^k|^2\frac{(m_T^2 - m_Z^2)^2(m_T^2 + 2m_Z^2)}{m_T^5} && \text{(B.15)}\\
\Gamma(T^- \to W^-\nu^k) &= \frac{1}{16\pi}|y^k|^2\frac{(m_T^2 - m_W^2)^2(m_T^2 + 2m_W^2)}{m_T^5} && \text{(B.16)}\\
\Gamma(T^3 \to W^+e^k) &= \Gamma(T^3 \to W^-\bar{e}^k) = \\
&= \frac{1}{32\pi}|y^k|^2\frac{(m_T^2 - m_W^2)^2(m_T^2 + 2m_W^2)}{m_T^5} && \text{(B.17)}
\end{aligned}$$

Now we only need to find the decay rate for $T^0 \to \nu^k Z$. From Eq. (B.1), the amplitude is given by

$$\mathcal{M} = -\frac{e\epsilon_T^j}{2s_W c_W}\left[\bar{u}_\nu^r(p_2)\gamma^\mu\gamma^5 u_T^s(p_1)\right]\epsilon_{\mu Z}^*(p_3) \tag{B.18}$$

so that

$$\begin{aligned}
\overline{|\mathcal{M}|^2} &= \frac{1}{2}\left(\frac{e\epsilon_T^j}{2s_W c_W}\right)^2 Tr\left[(\not{p}_1 - m_1)\gamma^\mu\gamma^5\not{p}_2\gamma^\nu\gamma^5\right]\left(-g_{\mu\nu} + \frac{p_{3\mu}p_{3\nu}}{m_Z^2}\right)\\
&= \frac{1}{2}\left(\frac{e\epsilon_T^j}{2s_W c_W}\right)^2 Tr\left[\not{p}_1\gamma^\mu\not{p}_2\gamma^\nu\right]\left(-g_{\mu\nu} + \frac{p_{3\mu}p_{3\nu}}{m_Z^2}\right) && \text{(B.19)}
\end{aligned}$$

which is equivalent to Eq. (B.3).

Therefore we can simply use the decay rate formula we have found previously and we will get

$$\Gamma(T^3 \to Z\nu_k) = \frac{1}{32\pi}|y_T^k|^2\frac{(m_T^2 - m_Z^2)^2(m_T^2 + 2m_Z^2)}{m_T^5} \tag{B.20}$$

B.2 $T \to f + H$

When the triplets are heavier than the Higgs particle, it can also decay to Higgs through Yukawa interaction

$$\mathcal{L} = -\frac{h}{\sqrt{2}}\left(\sqrt{2}y_T^j \bar{e}_R^j T_R^- + y_T^j \bar{\nu}^j T^3\right) \tag{B.21}$$

We now do a straightforward calculation for the amplitude analog with what we have done in the previous section and obtain

$$\begin{aligned}\overline{|\mathcal{M}|^2}_{T^-\to he} &= \frac{|y_T^j|^2}{8} Tr\left[(\not{p}_1 - m_T)(1-\gamma^5)\not{p}_2(1+\gamma^5)\right] \\ &= \frac{|y_T^j|^2}{4} Tr\left[\not{p}_1\not{p}_2\right] = |y_T^j|^2(p_1\cdot p_2) \\ &= |y_T^j|^2\frac{(m_T^2 - m_H^2)}{2} \end{aligned}\tag{B.22}$$

$$\begin{aligned}\overline{|\mathcal{M}|^2}_{T^-\to h\nu} &= \frac{|y_T^j|^2}{4} Tr\left[(\not{p}_1 - m_T)\not{p}_2\right] \\ &= |y_T^j|^2\frac{(m_T^2 - m_H^2)}{2}\end{aligned}\tag{B.23}$$

Therefore

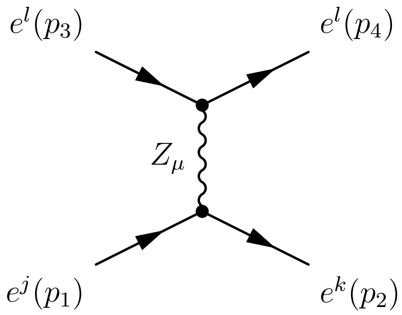
$$\Gamma(T\to he) = \Gamma(T\to h\nu) = \frac{1}{32\pi}|y_T^j|^2\frac{(m_T^2 - m_H^2)^2}{m_T^3} \tag{B.24}$$

B.3 $e^j \to e^k \bar{e}^l e^l$

The interaction responsible for this process comes from

$$\delta\mathcal{L} = -\frac{eZ_\mu}{2s_W c_W}\left[\bar{e}_L^j\gamma^\mu((2s_W^2 - 1)\delta^{jk} - 2\epsilon_T^j\epsilon_T^k)e_L^k + 2s_W^2\bar{e}_R^j\gamma^\mu e_R^j\right] \tag{B.25}$$

with diagram given by



The amplitude for this process is given by

$$\begin{aligned}\mathcal{M} &= \left(\frac{e}{2s_W c_W}\right)^2 (-2\epsilon_T^j\epsilon_T^k)\bar{u}_k^r(p_2)\gamma^\alpha\frac{(1-\gamma^5)}{2}u_j^{r'}(p_1)\frac{g_{\alpha\beta}-(k_\alpha k_\beta/m_Z^2)}{k^2-m_Z^2} \\ &\quad \bar{u}_l^s(p_4)\gamma^\beta\frac{(g_V+\gamma^5)}{2}u_l^{s'}(p_3) \qquad \text{(B.26)}\end{aligned}$$

where r, r', s, s' denotes the spin of each spinors, $k \equiv p_2 - p_1 = p_4 - p_3$ and $g_V = 4s_W^2 - 1$. If we take the square of Eq. (B.26) and averaged over all final spin we will get (in the limit $m_Z^2 \gg q^2$ and $m_{e^k} = m_{e^l} \sim 0$).

$$\begin{aligned}\overline{|\mathcal{M}|^2} &= \frac{1}{256}\left(\frac{e}{s_W c_W m_Z}\right)^4 (\epsilon_T^j\epsilon_T^k)^2 Tr\left[\not{p}_1\gamma^\alpha\not{p}_2\gamma^\beta(1-\gamma^5)\right] \\ &\quad Tr\left[\not{p}_3\gamma_\alpha\not{p}_4\gamma_\beta\left(g_V+\gamma^5\right)^2\right] \qquad \text{(B.27)}\end{aligned}$$

The trace product can be evaluated by first noticing that

$$\begin{aligned}Tr[\not{p}_1\gamma^\alpha\not{p}_2\gamma^\beta]Tr[\not{p}_3\gamma_\alpha\not{p}_4\gamma_\beta] &= 32[(p_1\cdot p_3)(p_2\cdot p_4)+(p_1\cdot p_4)(p_2\cdot p_3)] \\ Tr[\not{p}_1\gamma^\alpha\not{p}_2\gamma^\beta\gamma^5]Tr[\not{p}_3\gamma_\alpha\not{p}_4\gamma_\beta\gamma^5] &= 32[(p_1\cdot p_3)(p_2\cdot p_4)-(p_1\cdot p_4)(p_2\cdot p_3)] \\ Tr[\not{p}_1\gamma^\alpha\not{p}_2\gamma^\beta]Tr[\not{p}_3\gamma_\alpha\not{p}_4\gamma_\beta\gamma^5] &= 0 \qquad \text{(B.28)}\end{aligned}$$

Hence Eq. (B.27) becomes

$$\begin{aligned}\overline{|\mathcal{M}|^2} &= \frac{1}{8}\left(\frac{e}{s_W c_W m_Z}\right)^4 (\epsilon_T^j\epsilon_T^k)^2 \\ &\quad \left[(g_V-1)^2(p_1\cdot p_3)(p_2\cdot p_4)+(g_V+1)^2(p_1\cdot p_4)(p_2\cdot p_3)\right] \qquad \text{(B.29)}\end{aligned}$$

If we work in CM frame where the initial particle is at rest we will have

$$(p_1-p_3)^2 = (p_2+p_4)^2 \qquad \text{(B.30)}$$

$$(p_1\cdot p_3) = m_j E_3 \qquad \text{(B.31)}$$

$$\begin{aligned}2(p_2\cdot p_4) &= (p_2+p_4)^2 - p_2^2 - p_4^2 \\ &= (p_1-p_3)^2 = m_j(m_j-2E_3) \qquad \text{(B.32)}\end{aligned}$$

$$(p_1\cdot p_4) = m_j E_4 \qquad \text{(B.33)}$$

$$2(p_2\cdot p_3) = m_j(m_j-2E_4) \qquad \text{(B.34)}$$

Thus the amplitude becomes

$$\begin{aligned}\overline{|\mathcal{M}|^2} &= \frac{1}{16}\left(\frac{e}{s_W c_W m_Z}\right)^4 \left(\epsilon_T^j \epsilon_T^k\right)^2 \\ &\quad \Big[(g_V-1)^2 m_j^2 E_3(m_j-2E_3)+(g_V+1)^2 m_j^2 E_4(m_j-2E_4)\Big] \end{aligned} \tag{B.35}$$

Since we have a symmetry between p_3 and p_4 we can freely change $E_4 \to E_3$ so that the previous relation is a little bit simplified to

$$\overline{|\mathcal{M}|^2} = \frac{1}{8}\left(\frac{e}{s_W c_W m_Z}\right)^4 \left(\epsilon_T^j \epsilon_T^k\right)^2 (g_V^2+1) m_j^2 E_3(m_j-2E_3) \tag{B.36}$$

Having found the amplitude we can now proceed on evaluating the kinematics. This process is a 3-body decay and the decay rate is given by the generic formula[15]

$$d\Gamma = \frac{\overline{|\mathcal{M}|^2}}{2m_j}\left(\frac{d^3\mathbf{p}_2}{(2\pi)^3 2E_2}\right)\left(\frac{d^3\mathbf{p}_3}{(2\pi)^3 2E_3}\right)\left(\frac{d^3\mathbf{p}_4}{(2\pi)^3 2E_4}\right)(2\pi)^4\delta^4(p_1-p_2-p_3-p_4) \tag{B.37}$$

where

$$E_{2,3,4} = \mathbf{p}_{2,3,4} \tag{B.38}$$

integrating over $\mathbf{p}_2$ yield

$$d\Gamma = \frac{2\overline{|\mathcal{M}|^2}}{(4\pi)^5 m_j}\frac{d^3\mathbf{p}_3 d^3\mathbf{p}_4}{E_2 E_3 E_4}\delta(m_j - E_2 - E_3 - E_4) \tag{B.39}$$

where now

$$E_2 = (\mathbf{p}_3 + \mathbf{p}_4) \tag{B.40}$$

Next we carry the integration over $\mathbf{p}_3$. Setting the axis along p_4 (which is fixed, for the purposes of the $\mathbf{p}_3$ integration), we have

$$d^3\mathbf{p}_3 = E_3^2 dE_3 \sin\theta d\theta d\phi \tag{B.41}$$

the integration over ϕ gives 2π, while to carry out the integration over θ first notice that

$$E_2 = \sqrt{E_3^2 + E_4^2 + 2E_3E_4\cos\theta} \tag{B.42}$$

so that

$$\frac{dE_2}{d\theta} = \frac{-E_3E_4\sin\theta}{E_2} \tag{B.43}$$

then

$$\int_0^\pi \frac{\sin\theta d\theta}{E_2}\delta(m_j - E_2 - E_3 - E_4) = \int_{E_2^-}^{E_2^+} \frac{dE_2}{E_2 E_4}\delta(m_j - E_2 - E_3 - E_4) = \left\{ \begin{array}{l} \frac{1}{E_3 E_4}, \text{if} E_2^- < (m_j - E_3 - E_4) < E_2^+ \\ 0, \text{otherwise} \end{array} \right\} \tag{B.44}$$

where

$$E_2^\pm \equiv \sqrt{E_3^2 + E_4^2 \pm 2E_3E_4} = |E_3 \pm E_4| \tag{B.45}$$

The inequality on Eq. (B.44) is equal to

$$\frac{1}{2}\left[|E_3 - E_4| + E_2 + E_4\right] < \frac{1}{2}m_j < [E_3 + E_4] \tag{B.46}$$

which is equivalent to three inequalities

$$E_3 < \frac{m_j}{2} \tag{B.47}$$

$$E_4 < \frac{m_j}{2} \tag{B.48}$$

$$(E_2 + E_4) > \frac{m_j}{2} \tag{B.49}$$

these specifies the limits on the E_3 and E_4 integrals, i.e. E_3 goes from $\frac{m_j}{2} - E_4$ to $\frac{m_j}{2}$ and E_4 goes from 0 to $\frac{m_j}{2}$. The θ and ϕ integrals leave us with

$$d\Gamma = \frac{\langle|\mathcal{M}|^2\rangle}{(4\pi)^4 m_j} dE_3 \frac{d^3\mathbf{p}_4}{E_4^2} \tag{B.50}$$

We now plug the amplitude from Eq. (B.36) to the decay rate formula and we will obtain

$$\begin{aligned} d\Gamma &= \frac{1}{8}\left(\frac{e}{4\pi s_W c_W m_Z}\right)^4 (\epsilon_T^j \epsilon_T^k)^2 m_j (g_V^2 + 1)\frac{d^3\mathbf{p}_4}{E_4^2}\int_{\frac{m_j}{2}-E_4}^{\frac{m_j}{2}} dE_3 E_3 (m_j - 2E_3) \\ &= \frac{1}{8}\left(\frac{e}{4\pi s_W c_W m_Z}\right)^4 \left(\epsilon_T^j \epsilon_T^k\right)^2 m_j (g_V^2 + 1)\left(\frac{m_j}{2} - \frac{2}{3}E_4\right) d^3\mathbf{p}_4 \end{aligned} \tag{B.51}$$

Finally, we write

$$d^3\mathbf{p}_4 = 4\pi E_4^2 dE_4 \tag{B.52}$$

and we integrate over E_4 to obtain the total decay rate

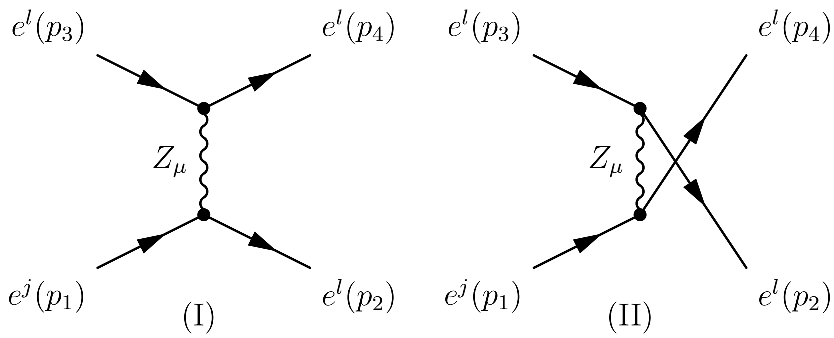
$$\begin{aligned}\Gamma &= \frac{\pi}{2}\left(\frac{e}{4\pi s_W c_W m_Z}\right)^4 \left(\epsilon_T^j \epsilon_T^k\right)^2 m_j (g_V^2+1)\int_0^{\frac{m_j}{2}}\left(\frac{m_j}{2}-\frac{2}{3}E_4\right)E_4^2 dE_4 \\ &= \frac{\pi}{192}\left(\frac{e}{4\pi s_W c_W m_Z}\right)^4 \left(\epsilon_T^j \epsilon_T^k\right)^2 m_j^5 (g_V^2+1) \\ &= \frac{m_j^5}{24(8\pi)^3 m_T^4}\left(y_T^j y_T^k\right)^2 (g_V^2+1) \qquad \text{(B.53)}\end{aligned}$$

where in the last line we have used

$$\epsilon_T^j = \frac{y_T^j v}{\sqrt{2} m_T} \quad ; \quad m_Z = \frac{ev}{2 s_W c_W} \qquad \text{(B.54)}$$

B.4 $e^j \to \bar{e}^l e^l e^l$

This decay is very similiar to the previous process we just discussed except now we have an additional cross diagram so the diagrams for the process are



(I) (II)

and the amplitude is given by

$$\mathcal{M} = \mathrm{I} - \mathrm{II} \qquad \text{(B.55)}$$

where I and II is given by

$$\mathrm{I} = \left(\frac{e}{2 s_W c_W}\right)^2 \frac{\epsilon_T^j \epsilon_T^k}{m_Z^2} \bar{u}_l^{s_2}(p_2)\gamma^\mu (g_V+\gamma^5) u_j^{s_1}(p_1) \bar{u}_l^{s_4}(p_4)\gamma_\mu \frac{1-\gamma^5}{2} u_l^{s_3}(p_3) \qquad \text{(B.56)}$$

$$\mathrm{II} = \left(\frac{e}{2 s_W c_W}\right)^2 \frac{\epsilon_T^j \epsilon_T^k}{m_Z^2} \bar{u}_l^{s_4}(p_4)\gamma^\mu (g_V+\gamma^5) u_j^{s_1}(p_1) \bar{u}_l^{s_2}(p_2)\gamma_\mu \frac{1-\gamma^5}{2} u_l^{s_3}(p_3) \qquad \text{(B.57)}$$

so the square of the amplitude is given by

$$\overline{|\mathcal{M}|^2} = \frac{1}{8}\sum_s \left[|\mathrm{I}|^2 + |\mathrm{II}|^2 - \mathrm{I}^*\mathrm{II} - \mathrm{II}^*\mathrm{I}\right] \tag{B.58}$$

The product of I and II can be obtained by using the trace theorem we have used in the previous section and we will have

$$\begin{aligned}
\frac{1}{8}\sum_s |\mathrm{I}|^2 &= \tfrac{1}{8}\left(\tfrac{e}{s_W c_W}\right)^4 \left(\tfrac{\epsilon_T^j \epsilon_T^k}{m_Z^2}\right)^2 \Big[(g_V-1)^2(p_1\cdot p_3)(p_2\cdot p_4) \\
&\qquad +(g_V+1)^2(p_1\cdot p_4)(p_2\cdot p_3)\Big] && \text{(B.59)}\\
\frac{1}{8}\sum_s |\mathrm{II}|^2 &= \tfrac{1}{8}\left(\tfrac{e}{s_W c_W}\right)^4 \left(\tfrac{\epsilon_T^j \epsilon_T^k}{m_Z^2}\right)^2 \Big[(g_V-1)^2(p_1\cdot p_3)(p_2\cdot p_4) \\
&\qquad +(g_V+1)^2(p_1\cdot p_2)(p_4\cdot p_3)\Big] && \text{(B.60)}\\
\frac{1}{8}|\mathrm{II}^*\mathrm{I}+\mathrm{I}^*\mathrm{II}| &= -\tfrac{1}{4}\left(\tfrac{e}{s_W c_W}\right)^4 \left(\tfrac{\epsilon_T^j \epsilon_T^k}{m_Z^2}\right)^2 \Big[(g_V-1)^2(p_1\cdot p_3)(p_2\cdot p_4)\Big] && \text{(B.61)}
\end{aligned}$$

Where in obtaining Eq. (B.61) we have used the following trace theorems[1]

$$\begin{aligned}
Tr\left[\not{p}_1\gamma_\mu\not{p}_2\gamma^\nu\not{p}_3\gamma^\mu\not{p}_4\gamma_\nu\right] &= -2Tr\left[\not{p}_1\not{p}_3\gamma^\nu\not{p}_2\not{p}_4\gamma_\nu\right] \\
&= -8\,(p_2\cdot p_4)\,Tr\left[\not{p}_1\not{p}_3\right] \\
&= -32\,(p_1\cdot p_3)\,(p_2\cdot p_4) && \text{(B.64)}
\end{aligned}$$

Thus we have

$$\begin{aligned}
\overline{|\mathcal{M}|^2} = \frac{1}{8}\left(\frac{e}{s_W c_W}\right)^4 \left(\frac{\epsilon_T^j \epsilon_T^k}{m_Z^2}\right)^2 &\big[2(g_V+1)^2(p_1\cdot p_4)(p_2\cdot p_3) \\
&+4(g_V-1)^2(p_1\cdot p_3)(p_2\cdot p_4)\big] && \text{(B.65)}
\end{aligned}$$

The integration over the momentum on the terms in this equation will give us the same result due to the simmetry between p_1, p_2 and p_3, therefore we can simply

[1] recall the following identity for gamma matrices

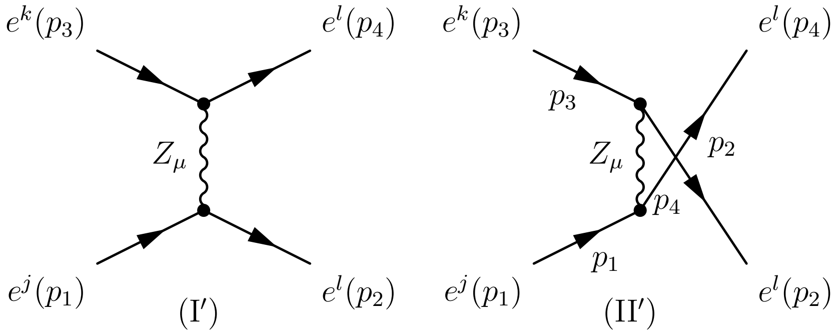
$$\begin{aligned}
\gamma^\mu\gamma_\nu\gamma_\rho\gamma_\mu &= 4g_{\mu\rho} && \text{(B.62)}\\
\gamma^\mu\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_\mu &= -2\gamma_\sigma\gamma_\rho\gamma_\nu && \text{(B.63)}
\end{aligned}$$

use the formula we obtain in the previous section to find the total decay rate

$$\Gamma = \frac{m_j^5}{48(8\pi)^3 m_T^4}\left(y_T^j y_T^k\right)^2\left[4(g_V-1)^2+2(g_V+1)^2\right] \tag{B.66}$$

B.5 $e^j \to \bar{e}^k e^l e^l$

This decay comes from the following diagram



which is very similiar to the diagram from $e^k \to e^l \bar{e}^l e^l$, except now both the vertex has coupling $\sim \epsilon_T^2$. Repeating the calculation we did in the previous section we get

$$\mathcal{M} = \mathrm{I}' - \mathrm{II}' \tag{B.67}$$

where

$$\mathrm{I}' = \left(\frac{e}{s_W c_W}\right)^2 \frac{\epsilon_T^j \epsilon_T^k \epsilon_T^l \epsilon_T^l}{m_Z^2} \bar{u}_l^{s_2}(p_2)\gamma^\mu \frac{1-\gamma^5}{2} u_j^{s_1}(p_1)\bar{u}_l^{s_4}(p_4)\gamma_\mu \frac{1-\gamma^5}{2} u_k^{s_3}(p_3) \tag{B.68}$$

$$\mathrm{II}' = \left(\frac{e}{s_W c_W}\right)^2 \frac{\epsilon_T^j \epsilon_T^k \epsilon_T^l \epsilon_T^l}{m_Z^2} \bar{u}_l^{s_4}(p_4)\gamma^\mu \frac{1-\gamma^5}{2} u_j^{s_1}(p_1)\bar{u}_l^{s_2}(p_2)\gamma_\mu \frac{1-\gamma^5}{2} u_k^{s_3}(p_3) \tag{B.69}$$

therefore

$$\frac{1}{8}\sum_s |\mathrm{I}'|^2 = 2\left(\frac{e}{s_W c_W}\right)^4 \left(\frac{\epsilon_T^j \epsilon_T^k \epsilon_T^l \epsilon_T^l}{m_Z^2}\right)^2 (p_1 \cdot p_3)\,(p_2 \cdot p_4) \quad \text{(B.70)}$$

$$\frac{1}{8}\sum_s |\mathrm{II}'|^2 = 2\left(\frac{e}{s_W c_W}\right)^4 \left(\frac{\epsilon_T^j \epsilon_T^k \epsilon_T^l \epsilon_T^l}{m_Z^2}\right)^2 (p_1 \cdot p_3)\,(p_2 \cdot p_4) \quad \text{(B.71)}$$

$$\frac{1}{8}\sum_s |\mathrm{II}'^*\mathrm{I}' + \mathrm{I}'^*\mathrm{II}'| = 4\left(\frac{e}{s_W c_W}\right)^4 \left(\frac{\epsilon_T^j \epsilon_T^k \epsilon_T^l \epsilon_T^l}{m_Z^2}\right)^2 (p_1 \cdot p_3)(p_2 \cdot p_4) \quad \text{(B.72)}$$

from there we have

$$\overline{|\mathcal{M}|}^2 = 8\left(\frac{e}{s_W c_W}\right)^4 \left(\frac{\epsilon_T^j \epsilon_T^k \epsilon_T^l \epsilon_T^l}{m_Z^2}\right)^2 (p_1 \cdot p_3)(p_2 \cdot p_4) \quad \text{(B.73)}$$

and finally the total decay rate is given by

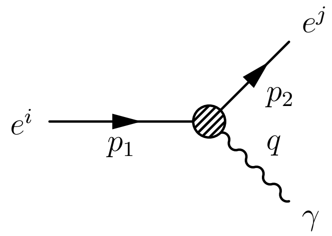
$$\begin{aligned} \Gamma &= \frac{\pi m_j^5}{6}\left(\frac{1}{4\pi}\right)^4 \left(\frac{e}{s_W c_W}\right)^4 \left(\frac{\epsilon_T^j \epsilon_T^k \epsilon_T^l \epsilon_T^l}{m_Z^2}\right)^2 \\ &= \frac{m_j^5 v^4}{3(8\pi)^3}\left(\frac{y_T^j y_T^k y_T^l y_T^l}{m_T^4}\right)^2 \end{aligned} \quad \text{(B.74)}$$

Appendix C

The $e^i \to e^j\gamma$ decay width

C.1 Notation and general form of the amplitude

In this section we are going to compute the decay width of a general $e^1 \to e^2\gamma$ decay[16]. We put p_1, p_2, m_1 and m_2 as the momentum of incoming and outgoing lepton respectively. In this case the photon's momentum will be equal to $q = p_1 - p_2$. In this calculation we restrict the photon to be on-shell ($q^2 = 0$) so this calculation will not be valid when $e^i \to e^j\gamma$ is only subprocess for bigger process. Since on most cases the second lepton will be much smaller than the first we can take $m_2 \to 0$. Furthermore we should have $p_1^2 = m_1^2$.



The general form of amplitude for this process can be written as

$$\mathcal{M} = \epsilon_\mu \bar{u}_j(p_2) J^\mu_{\text{em}} u_i(p_1) \tag{C.1}$$

The current is generally composed of γ^μ and combination of p_1^μ and p_2^μ therefore we can write it as

$$J^\mu_{\text{em}} = (p_1 + p_2)^\mu (A\gamma_L + B\gamma_R) + \gamma^\mu (C\gamma_L + D\gamma_R) + (p_1 - p_2)^\mu (E\gamma_L + F\gamma_R) \tag{C.2}$$

where A, B, C, D, E and F are numerical coefficients which should be obtained from evaluating loop diagrams. The matrix γ_L and γ_R are the projector of chirality defined as

$$\gamma_L = \frac{(1-\gamma^5)}{2} \quad ; \quad \gamma_R = \frac{(1+\gamma^5)}{2} \tag{C.3}$$

Since $\epsilon_\mu q^\mu = 0$, the third term on Eq. (C.2) should vanish. Moreover, gauge invariance requires $q_\mu \langle u_j | J^\mu_{\rm em} | u_i \rangle = 0$ therefore we should have

$$m_1^2(A\gamma_L + B\gamma_R) + m_1(C\gamma_R + D\gamma_L) = 0 \tag{C.4}$$

or

$$m_1 A = -D \tag{C.5}$$

$$m_1 B = -C \tag{C.6}$$

Using these relations the amplitude becomes

$$\mathcal{M} = \epsilon_\mu \bar{u}_j(p_2)\left[A\mathcal{J}^\mu_A + B\mathcal{J}^\mu_B\right] u_i(p_1) \tag{C.7}$$

where

$$\mathcal{J}^\mu_A = (p_1+p_2)^\mu\gamma_L - m_1\gamma^\mu\gamma_R \tag{C.8}$$

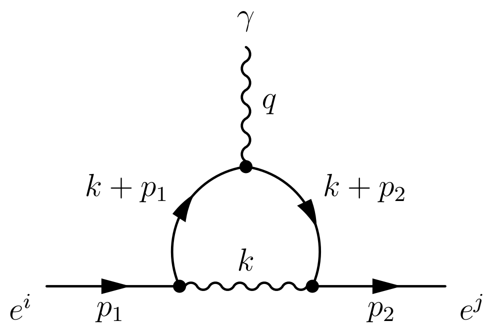
$$\mathcal{J}^\mu_B = (p_1+p_2)^\mu\gamma_R - m_1\gamma^\mu\gamma_L \tag{C.9}$$

Therefore we have only 2 independent parameter A and B which we have to calculate from loop diagrams. In addition, when we evaluate the loop we can concentrate only to terms proportional to $(p_1+p_2)^\mu$ since the terms proportional to γ^μ will give the same result in the end.

C.2 Basic Integrals

Before computing the amplitude from loop diagrams, we are going to show all the integral which we will frequently used later. Mainly there will be two loops in the main calculation, one where the photon couples to fermion and another where the photon couples to boson.

When the photon couples with fermion



We denote

$$D_B = k^2 - m_B^2 \tag{C.10}$$

$$D_{1F} = (k+p_1)^2 - m_F^2 \tag{C.11}$$

$$D_{2F} = (k+p_2)^2 - m_F^2 \tag{C.12}$$

and define

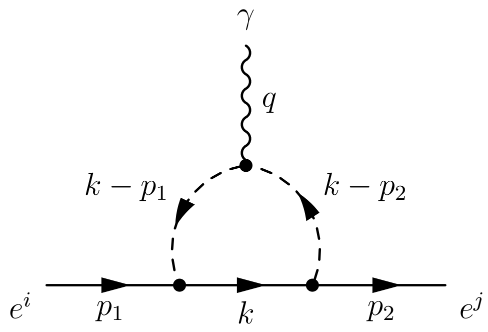
$$a = \int \frac{d^4k}{(2\pi)^4} \frac{1}{D_B D_{1F} D_{2F}} \tag{C.13}$$

$$c_1 p_1^\theta + c_2 p_2^\theta = \int \frac{d^4k}{(2\pi)^4} \frac{k^\theta}{D_B D_{1F} D_{2F}} \tag{C.14}$$

$$d_1 p_1^\theta p_1^\psi + d_2 p_2^\theta p_2^\psi + f\left(p_1^\theta p_2^\psi + p_2^\theta p_1^\psi\right) + x g^{\theta\psi} = \int \frac{d^4k}{(2\pi)^4} \frac{k^\theta k^\psi}{D_B D_{1F} D_{2F}} \tag{C.15}$$

On the calculation of the integration we will eventually shift the momentum $k \to k + px_1 + px_2$ where x_1 and x_2 is the parameter used when we do Feynman tricks to separate the denominator. Because of that the integration can be expressed as p_1 and p_2. Moreover, in this calculation we have a divergent part ($xg^{\theta\psi}$) which will cancel out with two-point integrals.

When the photon couples with bosons



we denote

$$D_{1B} = (k - p_1)^2 - m_B^2 \tag{C.16}$$

$$D_{2B} = (k - p_2)^2 - m_B^2 \tag{C.17}$$

$$D_F = k^2 - m_F^2 \tag{C.18}$$

and define

$$\bar{a} = \int \frac{d^4k}{(2\pi)^4} \frac{1}{D_{1B} D_{2B} D_F} \tag{C.19}$$

$$\bar{c}_1 p_1^\theta + \bar{c}_2 p_2^\theta = \int \frac{d^4k}{(2\pi)^4} \frac{k^\theta}{D_{1B} D_{2B} D_F} \tag{C.20}$$

$$\bar{d}_1 p_1^\theta p_1^\psi + \bar{d}_2 p_2^\theta p_2^\psi + \bar{f}\left(p_1^\theta p_2^\psi + p_2^\theta p_1^\psi\right) + \bar{x} g^{\theta\psi} = \int \frac{d^4k}{(2\pi)^4} \frac{k^\theta k^\psi}{D_{1B} D_{2B} D_F} \tag{C.21}$$

If we use the approximation $m_i^2 = m_j^2 = 0$ together with $q^2 = 0$, then the integral may be computed and we get the following result.

$$a = \frac{i}{16\pi^2 m_B^2}\left[\frac{-1}{t-1} + \frac{\ln t}{(t-1)^2}\right] \tag{C.22}$$

$$c_1 = c_2 \equiv c = \frac{i}{16\pi^2 m_B^2}\left[\frac{t-3}{4(t-1)^2} + \frac{\ln t}{2(t-1)^3}\right] \tag{C.23}$$

$$d_1 = d_2 = 2f \equiv d = \frac{i}{16\pi^2 m_B^2}\left[\frac{-2t^2+7t-11}{18(t-1)^3} + \frac{\ln t}{3(t-1)^4}\right] \tag{C.24}$$

$$\bar{a} = \frac{i}{16\pi^2 m_B^2}\left[\frac{1}{t-1} - \frac{t\ln t}{(t-1)^2}\right] \tag{C.25}$$

$$\bar{c}_1 = \bar{c}_2 \equiv \bar{c} = \frac{i}{16\pi^2 m_B^2}\left[\frac{3t-1}{4(t-1)^2} - \frac{t^2\ln t}{2(t-1)^3}\right] \tag{C.26}$$

$$\bar{d}_1 = \bar{d}_2 = 2\bar{f} \equiv \bar{d} = \frac{i}{16\pi^2 m_B^2}\left[\frac{11t^2-7t+2}{18(t-1)^3} - \frac{t^3\ln t}{3(t-1)^4}\right] \tag{C.27}$$

where $t = m_F^2/m_B^2$.

C.3 The General Interaction and Feynman Rules

First we are going to calculate this decay for general interactions then we will apply it for our theory. We assume the leptons to have interaction with some fermions with charges 0 or −1. In our theory the fermions can be neutrino, lepton, triplet or singlet fermion.

We write the general interaction as the following

$$\mathcal{L}_{\mathcal{W}} = \sum_i \bar{e}^i \left[\left(g_L^i \gamma^\mu \gamma_L + g_R^i \gamma^\mu \gamma_R \right) W_\mu^- + \left(f_L^i \gamma_L + f_R^i \gamma_R \right) \phi_W^- \right] F + h.c. \tag{C.28}$$

$$\mathcal{L}_{\mathcal{Z}} = \sum_i \bar{e}^i \left[(g_L'^i + \gamma^\mu \gamma_L + g_R'^i \gamma^\mu \gamma_R) Z_\mu + (f_L'^i \gamma_L + f_R'^i \gamma_R) \phi_Z^0 \right] N + h.c. \tag{C.29}$$

$$\mathcal{L}_{\mathcal{H}} = \sum_i \bar{e}^i \left[\left(h_L^i \gamma_L + h_R^i \gamma_R \right) h \right] N + h.c. \tag{C.30}$$

with F and N as the mediating fermion with charges −1 and 0 respectively and ϕ_W and ϕ_Z are the corresponding goldstone bosons for W_μ and Z_μ respectively.

As for the rest of the vertices, the Feynman rules with respect to the diagram shown in Fig. C.3 are given by[8, 10]

$$\text{(a)} = -ie[(k_1 - k_2)_\lambda g_{\mu\nu} + (k_2 - k_3) g_{\nu\lambda} + (k_3 - k_1)_\nu g_{\lambda\mu}] \tag{C.31}$$

$$\text{(b)} = ie(k_1 + k_2)^\mu \tag{C.32}$$

$$\text{(c)} = iem_W g_{\mu\nu} \tag{C.33}$$

C.4 Self energy diagrams

First we are going to make comments on self energy diagrams that may give rise to $e^1 \to e^2\gamma$ decay. In this case the photon may couple to the incoming or the outgoing fermion. However this kind of diagrams should be proportional to $\bar{u}(p_1)\gamma^\mu u(p_2)$, thus the result from this diagram should combine with the result

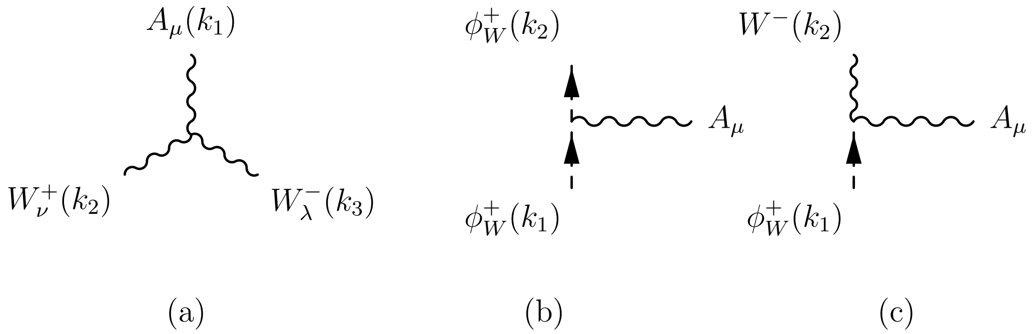


Figure C.1: Feynman diagrams for the vertices

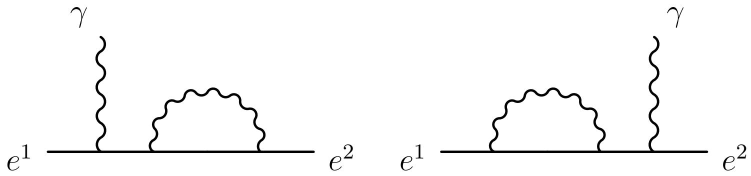


Figure C.2: Self energy diagrams for $e^1 \to e^2\gamma$ decay

from other diagram which proportional to γ^μ. We can safely ignore this part since we are going to concentrate only to terms which proportional to $(p_1 + p_2)^\mu$.

Also note that these diagrams may contain infinities, however those infinities will cancel automatically with the infinities in Eq. (C.15) as we have mentioned before.

C.5 The Loop Diagrams

All the loop diagrams that give rise to the decay are given in Fig. C.5. We denote T_X^G as the amplitude of each loop with fermion X and boson G. After a tedious but straightforward calculation using Dirac algebra, the amplitudes for each loops can be computed and expresed easily using the function given in Eq. (C.22)-(C.27).

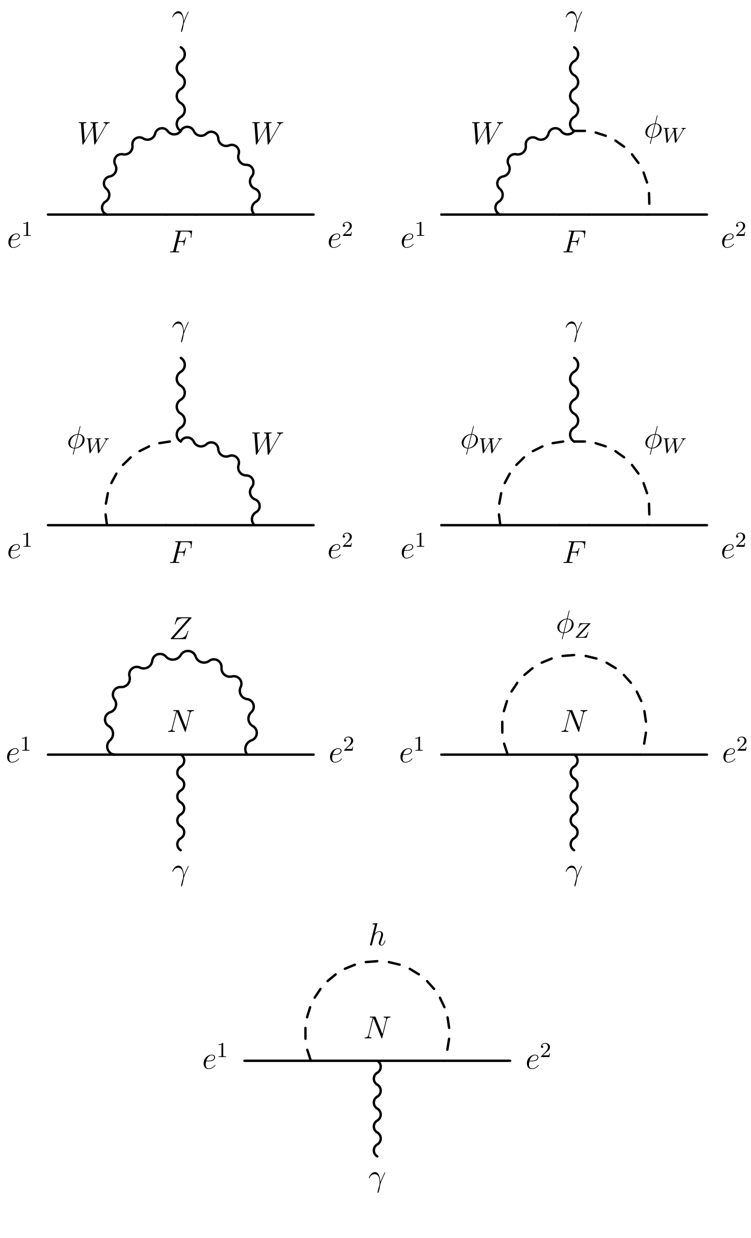


Figure C.3: Loop diagrams for $e^1 \to e^2\gamma$ decay

With 'tHooft-Feynman gauge, the result is given as following

$$\begin{aligned}
T_F^W &= e\epsilon_\mu^*(q)\bar{u}_2(p_2)(p_1+p_2)^\mu\big[(g_L^{*1}g_L^2\gamma_R + g_R^{*1}g_R^2\gamma_L)m_1(\bar{c}+3\bar{d}) \\
&+ (g_L^{*1}g_R^2\gamma_L + g_R^{*1}g_L^2\gamma_R)m_F(-6\bar{c})\big]u_1(p_1) + \ldots \qquad \text{(C.34)} \\
T_F^{W,\phi_W} &= e\epsilon_\mu^*(q)\bar{u}_2(p_2)(p_1+p_2)^\mu\big[(g_L^{*1}f_L^2\gamma_R + g_R^{*1}f_R^2\gamma_L)m_W(-\bar{c})\big]u_1(p_1) \\
&+ \ldots \qquad \text{(C.35)} \\
T_F^{\phi_W,W} &= e\epsilon_\mu^*(q)\bar{u}_2(p_2)(p_1+p_2)^\mu\big[(f_L^{*1}g_L^2\gamma_R + f_R^{*1}g_R^2\gamma_L)m_W(\bar{c})\big]u_1(p_1) \\
&+ \ldots \qquad \text{(C.36)} \\
T_F^{\phi_W} &= e\epsilon_\mu^*(q)\bar{u}_2(p_2)(p_1+p_2)^\mu\big[(f_R^{*1}f_R^1\gamma_R + f_L^{*1}f_L^1\gamma_L)m_1\left(\frac{3}{2}\bar{d}-\bar{c}\right) \\
&+ (f_L^{*1}f_R^2\gamma_R + f_R^{*1}f_L^2\gamma_L)m_F(2\bar{c}-\bar{a})\big]u_1(p_1) + \ldots \qquad \text{(C.37)} \\
T_N^Z &= -e\epsilon_\mu^*(q)\bar{u}_2(p_2)(p_1+p_2)^\mu\big[(g_L'^{*1}g_L'^2\gamma_R + g_R'^{*1}g_R'^2\gamma_L)m_1(6c+3d+2a) + \\
&+ (g_L'^{*1}g_R'^2\gamma_L + g_R'^{*1}g_L'^2\gamma_R)m_N(-8c-4a)\big]u_1(p_1) + \ldots \qquad \text{(C.38)} \\
T_N^{\phi_Z} &= -e\epsilon_\mu^*(q)\bar{u}_2(p_2)(p_1+p_2)^\mu\big[(f_L'^{*1}f_L'^2\gamma_L + f_R'^{*1}f_R'^2\gamma_R)m_1\left(\frac{3}{2}d+c\right) \\
&+ (f_L'^{*1}f_R'^2\gamma_R + f_R'^{*1}f_L'^2\gamma_L)m_F(2c)\big]u_1(p_1) + \ldots \qquad \text{(C.39)} \\
T_N^h &= -e\epsilon_\mu^*(q)\bar{u}_2(p_2)(p_1+p_2)^\mu\big[(h_L^{*1}h_L^2\gamma_L + h_R^{*1}h_R^2\gamma_R)m_1\left(\frac{3}{2}d+c\right) \\
&+ (h_L^{*1}h_R^2\gamma_R + h_R^{*1}h_L^2\gamma_L)m_F(2c)\big]u_1(p_1) + \ldots \qquad \text{(C.40)}
\end{aligned}$$

where "..." stands for the part proportional to γ^μ which we ignored.

We can now find the amplitude for the $SU(5)$ theory using the previous formula by substituting $F \to T^-, e^i$ and $N \to \nu^i, T^0, S$. We then use the appropriate couplings with respect to all the interactions given in Eq. (5.49). After some algebra we will get the following result at $\mathcal{O}\left(\left(\frac{y_T^i v}{m_T}\right)^2\right)$

$$\begin{aligned}
T_{\nu^i} &= T_{\nu^i}^{\phi^-} + T_{\nu^i}^{\phi^-,W^-} + T_{\nu^i}^{W^-,\phi^-} + T_{\nu^i}^{W^-} \\
&= \frac{g^2 e}{32\pi^2 m_W^2}\bar{u}_2(p_2)m_1(p_1+p_2)^\mu\gamma_R\left[\epsilon_T^2\epsilon_T^1 F_1(x_{\nu^i}) - \frac{1}{2}\epsilon_S^{*2}\epsilon_S^1 F_2(x_{\nu^i})\right]u_1(p_1) \qquad \text{(C.41)}
\end{aligned}$$

$$\begin{aligned}
T_{T^3} &= T_{T^3}^{\phi^-} + T_{T^3}^{\phi^-,W^-} + T_{T^3}^{W^-,\phi^-} + T_{T^3}^{W^-} \\
&= \frac{g^2 e}{32\pi^2 m_W^2}\bar{u}_2(p_2)m_1(p_1+p_2)^\mu\gamma_R\left[\epsilon_T^2\epsilon_T^1 F_3(x_T)\right]u_1(p_1) \qquad \text{(C.42)}
\end{aligned}$$

$$\begin{aligned}
T_S &= T_S^{\phi^-} + T_S^{\phi^-,W^-} + T_S^{W^-,\phi^-} T_S^{W^-} \\
&= \frac{g^2 e}{32\pi^2 m_W^2} \bar{u}_2(p_2) m_1 (p_1+p_2)^\mu \gamma_R \left[\epsilon_S^{*2}\epsilon_S^1 F_4(x_S)\right] u_1(p_1) \qquad (C.43)\\
T_{T^-} &= T_{T^-}^Z + T_{T^-}^h + T_{T^-}^\eta \\
&= \frac{g^2 e}{32\pi^2 m_W^2} \bar{u}_2(p_2) m_1 (p_1+p_2)^\mu \gamma_R \left[\epsilon_T^2\epsilon_T^1 (F_5(y_T) + F_6(z_T)\right] u_1(p_1) \qquad (C.44)
\end{aligned}$$

$$\begin{aligned}
T_{e^i} &= T_{e^i}^Z + T_{e^i}^h + T_{e^i}^\eta \\
&= \frac{g^2 e}{32\pi^2 m_W^2} \bar{u}_2(p_2) m_1 (p_1+p_2)^\mu \gamma_R \left[\epsilon_T^2\epsilon_T^1 G(y_{e^i}, z_{e^i})\right] u_1(p_1) \qquad (C.45)
\end{aligned}$$

where $x_{\nu^i} = \frac{m_{\nu^i}^2}{m_W^2}, x_T = \frac{m_T^2}{m_W^2}, x_S = \frac{m_S^2}{m_W^2}, y_{e^i} = \frac{m_{e^i}^2}{m_Z^2}, z_{e^i} = \frac{m_{e^i}^2}{m_h^2}, y_T = \frac{m_T^2}{m_Z^2}, z_T = \frac{m_T^2}{m_h^2}$ and $F_i(x)$ and $G(x)$ are the following functions

$$F_1(x) = \frac{31x^3 - 57x^2 + 33x - 7 - 6x^2(3x-1)\ln(x)}{12(x-1)^4} \qquad (C.46)$$

$$F_2(x) = \frac{9x^3 - 21x^2 + 15x - 3 - 6x^2(x-1)\ln(x)}{12(x-1)^4} \qquad (C.47)$$

$$F_3(x) = \frac{8x^4 - 71x^3 + 162x^2 - 125x + 26 - (24x^3 - 12x^2 - 6x)\ln(x)}{12(x-1)^4} \qquad (C.48)$$

$$F_4(x) = \frac{8x^4 + 37x^3 - 90x^2 + 53x - 8 - 42x^3\ln(x)}{12(x-1)^4} \qquad (C.49)$$

$$F_5(x) = \frac{-7x^4 - 2x^3 + 3x^2 + 46x - 40 - 18x(3x-4)\ln(x)}{12(x-1)^4} \qquad (C.50)$$

$$F_6(x) = \frac{-7x^4 + 36x^3 - 45x^2 + 16x - 6x(3x-2)\ln(x)}{12(x-1)^4} \qquad (C.51)$$

$$\begin{aligned}
G(y_{e^i}, z_{e^i}) &= \delta^{i2} \left[8\left(\frac{1}{2} - c_W^2\right)\frac{-5y_{e^i}^3 + 9y_{e^i} - 4 + 6(2y_{e^i}-1)y_{e^i}\ln(y_{e^i})}{24(y_{e^i}-1)^4}\right] \\
&+\delta^{i1} \left[z_{l^i}\frac{7z_{l^i}^3 - 36z_{l^i}^2 + 45z_{l^i} - 16 + 6(3z_{l^i}-2)\ln(x)}{8(z_{l^i}-1)^4}\right. \\
&+8\left(\frac{1}{2} - c_W^2\right)\frac{-5y_{e^i}^3 + 9y_{e^i} - 4 + 6(2y_{e^i}-1)y_{e^i}\ln(y_{e^i})}{24(y_{e^i}-1)^4} \\
&-8\left(1 - c_W^2\right)\frac{2(-y_{e^i}^2 + 1 + 2y_{e^i}\ln(y_{e^i})}{4(y_{e^i}-1)^3} \\
&\left.-y_{e^i}\frac{-5y_{e^i}^3 + 24y_{e^i}^2 - 39y_{e^i} + 20 - 6(y_{e^i}-2)\ln(y_{e^i})}{6(y_{e^i}-1)^4}\right] \qquad (C.52)
\end{aligned}$$

Since $x_{\nu^i}, y_{e^i}, z_{e^i} \ll 1$, it is a good approximation to take those terms to zero and take only the leading order. We would have then

$$F_1(x_{\nu^i}) \simeq \frac{-7}{12} \quad (C.53)$$

$$F_2(x_{\nu^i}) \simeq \frac{-3}{12} \quad (C.54)$$

$$G(y_{e^i}, z_{e^i}) \simeq -1,64 \quad (C.55)$$

with respect to Eq. (C.7) we have the parameter A and B as given below

$$\begin{aligned} A &= \frac{m_1 g^2 e}{32\pi^2 m_W^2} \left[\epsilon_T^2 \epsilon_T^1 \left(\frac{-7}{12} - 1,64 + F_3(x_T) + F_5(y_T) + F_6(z_T)\right)\right. \\ &\quad \left. + \epsilon_S^{*2} \epsilon_S^1 \left(\frac{-3}{12} + F_4(x_S)\right)\right] \\ &= \frac{m_1 g^2 e}{32\pi^2 m_W^2} \left[\epsilon_T^2 \epsilon_T^1 \left(\frac{-13}{12} - 1,64 + \mathcal{A}(x_T) + \mathcal{B}(y_T) + \mathcal{C}(z_T)\right)\right. \\ &\quad \left. + \epsilon_S^{*2} \epsilon_S^1 \left(\frac{5}{12} + \mathcal{D}(x_S)\right)\right] \end{aligned} \quad (C.56)$$

$$B = 0 \quad (C.57)$$

with

$$\mathcal{A}(x) = \frac{-39x^3 + 114x^2 - 93x - 6 - (24x^3 - 12x^2 - 6x)\ln(x)}{12(x-1)^4} \quad (C.58)$$

$$\mathcal{B}(x) = \frac{-30x^3 + 45x^2 + 18x - 33 - 18x(3x-4)\ln(x)}{12(x-1)^4} \quad (C.59)$$

$$\mathcal{C}(x) = \frac{8x^3 - 3x^2 - 12x + 7 - 6x(3x-1)\ln(x)}{12(x-1)^4} \quad (C.60)$$

$$\mathcal{D}(x) = \frac{69x^3 - 138x^2 + 85x - 16 - 42x^3\ln(x)}{12(x-1)^4} \quad (C.61)$$

C.6 Decay Rate

Using all the result we have obtained from the previous section the amplitude can be written as

$$\mathcal{M} = \epsilon_\mu \bar{u}_j(p_2)[A\mathcal{J}_\mathcal{A}^\mu + \mathcal{B}\mathcal{J}_B^\mu]u_i(p_1) \quad (C.62)$$

where A and B are given in Eq. (C.56) and Eq. (C.57). To calculate the decay rate we need to calculate the square of amplitude $\overline{|\mathcal{M}|^2}$. First we are going to use

Gordon identity

$$i\sigma^{\mu\nu}q_\nu = (p_1+p_2)^\mu + \not{p}_2\gamma^\mu + \gamma^\mu\not{p}_1 \qquad \text{(C.63)}$$

$$\sigma^{\mu\nu} = \frac{i}{2}\left[\gamma^\mu,\gamma^\nu\right] \qquad \text{(C.64)}$$

to change the (p_1+p_2) term in favour of q. Using that identity the amplitude can be written as

$$\mathcal{M} = \epsilon_\mu \bar{u}_j(p_2)\left[i\sigma^{\mu\nu}q_\nu\left(A\gamma_L + B\gamma_R\right)\right]u_i(p_1) \qquad \text{(C.65)}$$

The square of the amplitude can now be calculated more easily, we will have it given by

$$\begin{aligned}\overline{|\mathcal{M}|^2} &= \sum_{\text{spins}} \mathcal{M}^{*\mu}\mathcal{M}_\mu \\ &= Tr\left[(\not{p}_1 - m_i)(A^*\gamma_R + B^*\gamma_L)(q_\alpha\sigma^{\mu\alpha})\not{p}_2\sigma^\beta_\mu q_\beta(A\gamma_L + B\gamma_R)\right]\end{aligned} \qquad \text{(C.66)}$$

The term inside the trace can be simplified using the relation

$$\sigma^{\mu\alpha}\sigma^\beta_\mu = -\delta^{\alpha\beta} - 2\gamma^\alpha\gamma^\beta \qquad \text{(C.67)}$$

$$\sigma^{\mu\alpha}\gamma^\gamma\sigma^\beta_\mu = \delta^{\alpha\gamma}\gamma^\beta + \delta^{\gamma\beta}\gamma^\alpha + \frac{1}{2}\left(\gamma^\beta\gamma^\gamma\gamma^\alpha + \gamma^\alpha\gamma^\gamma\gamma^\beta\right) \qquad \text{(C.68)}$$

so that

$$q_\alpha\sigma^{\mu\alpha}\sigma^\beta_\mu q_\beta = 0 \qquad \text{(C.69)}$$

$$\gamma_R\sigma^{\mu\alpha}\gamma^\gamma\sigma^\beta_\mu\gamma_R = 0 \qquad \text{(C.70)}$$

$$\gamma_L q_\alpha\sigma^{\mu\alpha}\gamma^\gamma\sigma^\beta_\mu q_\beta\gamma_R = \not{q}\gamma^\gamma\not{q}\gamma_R \qquad \text{(C.71)}$$

$$\gamma_R q_\alpha\sigma^{\mu\alpha}\gamma^\gamma\sigma^\beta_\mu q_\beta\gamma_L = \not{q}\gamma^\gamma\not{q}\gamma_L \qquad \text{(C.72)}$$

and Eq. (C.66) becomes

$$\begin{aligned}\overline{|\mathcal{M}|^2} &= Tr\left[\not{p}_1\not{q}\not{p}_2\not{q}(|A|^2\gamma_L + |B|^2\gamma_R)\right] \\ &= 8(|A|^2+|B|^2)(p_1\cdot q)(p_2\cdot q) \\ &= 2m_1^4(|A|^2+|B|^2)\end{aligned} \qquad \text{(C.73)}$$

where we have used

$$2(p_1 \cdot q) = 2(p_2 \cdot q) = m_1^2 \tag{C.74}$$

Finally we can obtain the total decay rate from the generic formula of 2 body decay

$$\begin{aligned}\Gamma &= \frac{1}{16\pi}\overline{|\mathcal{M}|^2}\frac{|\mathbf{p}_2|}{m_1^2} \\ &= \frac{1}{16\pi}m_1^3\left(|A|^2+|B|^2\right)\end{aligned} \tag{C.75}$$

where

$$|\mathbf{p}_2| = \frac{m_1^2 - m_2^2}{2m_1} \tag{C.76}$$

Appendix D

Triplet Mass Splitting

At 1-loop order the inverse T fermion propagator is

$$\not{p} - m_T + \Sigma(p) \tag{D.1}$$

so that parametrizing the 1-loop contribution

$$\Sigma(p) = A(p^2)\not{p} + B(p^2) \tag{D.2}$$

one gets for the pole mass

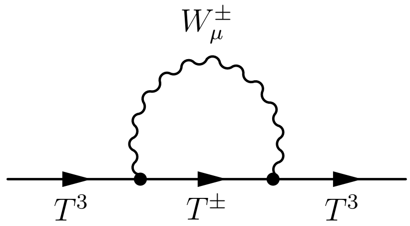
$$m_{T,pole} = \frac{m_T - B(m_{T,pole}^2)}{1 + A(m_{T,pole}^2)} \tag{D.3}$$

At leading order, the difference between 1-loop and tree order is thus

$$\delta m_{T,pole} \approx -(B(m_T^2) + m_T A(m_T^2)) \tag{D.4}$$

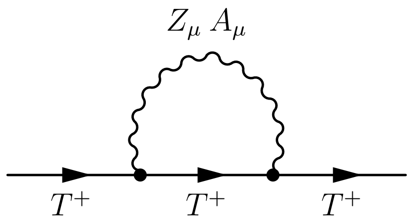
As we will see this contribution is different for $T^{\pm}$ and T^3. For $T^{\pm}$ the loops can be made of W, Z and A boson while for T^3 there can only be loop from W boson, therefore their mass should be splitted. In calculating this split we have ignored the contribution from Yukawa interactions since it is typically small compared to those from the gauge interactions.

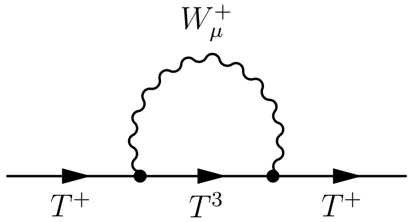
We will obtain the parameter $A(p^2)$ and $B(p^2)$ from the loop. For T^3 we will have



$$\Sigma_3(p) = 2i\left(\frac{e}{s_W}\right)^2 \int \frac{d^dk}{(2\pi)^d}\mu^{4-d}\frac{\gamma^\alpha[\not{k}+\not{p}+m_T]\gamma^\beta}{[(k+p)^2-m_T^2]\,[k^2-m_W^2]}\left[g_{\alpha\beta}-\frac{(1-\xi_W)k_\alpha k_\beta}{k^2-\xi_W m_W^2}\right] \tag{D.5}$$

and for T^+ we will have





$$\begin{aligned}\Sigma_+(p) &= i\left(\frac{e}{s_W}\right)^2 \int \frac{d^dk}{(2\pi)^d}\mu^{4-d}\frac{\gamma^\alpha[\not{k}+\not{p}+m_T]\gamma^\beta}{[(k+p)^2-m_T^2]\,[k^2-m_W^2]}\left[g_{\alpha\beta}-\frac{(1-\xi_W)k_\alpha k_\beta}{k^2-\xi_W m_W^2}\right]\\ &+ i\left(\frac{ec_W}{s_W}\right)^2 \int \frac{d^dk}{(2\pi)^d}\mu^{4-d}\frac{\gamma^\alpha[\not{k}+\not{p}+m_T]\gamma^\beta}{[(k+p)^2-m_T^2]\,[k^2-m_Z^2]}\left[g_{\alpha\beta}-\frac{(1-\xi_Z)k_\alpha k_\beta}{k^2-\xi_Z m_Z^2}\right]\\ &+ i(e)^2 \int \frac{d^dk}{(2\pi)^d}\mu^{4-d}\frac{\gamma^\alpha[\not{k}+\not{p}+m_T]\gamma^\beta}{[(k+p)^2-m_T^2][k^2]]}\left[g_{\alpha\beta}-\frac{(1-\xi_A)k_\alpha k_\beta}{k^2}\right]\end{aligned} \tag{D.6}$$

As the gauge, we can choose $\xi = 1$ and the corresponding loop from the goldstone boson will be too small to have an effect to our calculation so that we can neglect it[1].

We can now proceed to the calculation of each loops. First we denote

$$ia(m_B) + x_1 = \int \frac{d^dk}{(2\pi)^d}\frac{1}{[(k+p)^2-m_T^2][k^2-m_B^2]} \tag{D.7}$$

$$ib(m_B)p^\theta + x_2 p^\theta = \int \frac{d^dk}{(2\pi)^d}\frac{k^\theta}{[(k+p)^2-m_T^2][k^2-m_B^2]} \tag{D.8}$$

[1]the vertex of goldstone boson and two T's is proportional to $y_T^i \epsilon_T^i$

Where x_1 and x_2 are the part which are either divergent or independent of m_B, they will automatically cancel when we substract Σ_3 from Σ_+. Afterwards the integration can be computed and after taking $d \to 4+\epsilon$ and $p^2 = m_T^2$ we will get the following result ($m_T \geq m_B$)

$$16\pi^2 a(m_B) = -\frac{m_B^2}{2m_T^2}\ln\left(\frac{m_T^2}{m_B^2}\right) + \frac{m_B^2}{m_T^2}\sqrt{4\left(\frac{m_T^2}{m_B^2}\right)-1}\arctan\sqrt{4\left(\frac{m_T^2}{m_B^2}\right)-1} \tag{D.9}$$

$$\begin{aligned}16\pi^2 b(m_B) &= \left[\left(\frac{m_B^2}{2m_T^2}\right)^2 - \frac{m_B^2}{2m_T^2}\right]\ln\left(\frac{m_T^2}{m_B^2}\right) \\ &+ 2\left(\frac{m_B^2}{2m_T^2}\right)^2\sqrt{4\left(\frac{m_T^2}{m_B^2}\right)-1}\arctan\sqrt{4\left(\frac{m_T^2}{m_B^2}\right)-1} - \frac{m_B^2}{2m_T^2}\end{aligned} \tag{D.10}$$

Using this notation, $\Sigma(p)$ can be expressed easily, we will have

$$\Sigma_3(p) = -2\left(\frac{e}{s_W}\right)^2\left[4m_T a(m_W^2) - 2(a(m_W^2)+b(m_W^2))\not{p}\right] \tag{D.11}$$

$$\begin{aligned}\Sigma_+(p) &= -\left(\frac{e}{s_W}\right)^2\left[4m_T a(m_W^2) - 2(a(m_W^2)+b(m_W^2))\not{p}\right] \\ &- \left(\frac{ec_W}{s_W}\right)^2\left[4m_T a(m_Z^2) - 2(a(m_Z^2)+b(m_Z^2))\not{p}\right]\end{aligned} \tag{D.12}$$

The mass difference can be computed by using Eq. (D.4) and after a little bit of algebra we find

$$\begin{aligned}\Delta m_T &= \delta m^3_{T,pole} - \delta m^+_{T,pole} \\ &= \frac{\alpha_2}{2\pi}\frac{m_W^2}{m_T}\left[f\left(\frac{m_T^2}{m_Z^2}\right) - f\left(\frac{m_T^2}{m_W^2}\right)\right]\end{aligned} \tag{D.13}$$

where

$$f(y) = \frac{1}{4y}\ln y - \left(1+\frac{1}{2y}\right)\sqrt{4y-1}\arctan\sqrt{4y-1} \tag{D.14}$$

Appendix E

Neutrino Masses with Triplet and Singlet

The mass Lagrangian for Triplet, Singlet and left-handed neutrinos is given by (from Eq. (5.20) and Eq. (5.22))

$$\mathcal{L}_{\text{mass}} = -\frac{1}{2}\vec{\nu}_R^T C \mathbf{M}_R \vec{\nu}_R + \langle \phi^0 \rangle \vec{\nu}_R^T C \mathbf{Y}_\nu \vec{\nu}_L \tag{E.1}$$

where the states are defined as following

$$\vec{\nu}_R \equiv \begin{pmatrix} T^3 \\ S \\ 0 \end{pmatrix} \quad ; \quad \vec{\nu}_L \equiv \begin{pmatrix} \nu_L^e \\ \nu_L^\mu \\ \nu_L^\tau \end{pmatrix} \tag{E.2}$$

and $\mathbf{M}_R$ and $\mathbf{Y}_\nu$ is the mass matrix for T and S and Yukawa coupling matrix respectively. They are defined as the following

$$\mathbf{M}_R = \begin{pmatrix} M_T & 0 & 0 \\ 0 & M_S & 0 \\ 0 & 0 & \infty \end{pmatrix} \quad ; \quad \mathbf{Y}_\nu = \begin{pmatrix} y_T^e & y_T^\mu & y_T^\tau \\ y_S^e & y_S^\mu & y_S^\tau \\ 0 & 0 & 0 \end{pmatrix} \tag{E.3}$$

Integrating $\vec{\nu}_R$ out from Eq. (E.1) we obtain the following effective Lagrangian for $\vec{\nu}_L$

$$\delta\mathcal{L} = -\frac{1}{2}\vec{\nu}_L^T C \mathbf{M}_L \vec{\nu}_L + h.c. \tag{E.4}$$

with

$$\mathbf{M}_L = \langle \phi^0 \rangle^2 \mathbf{Y}_\nu^T \mathbf{M}_R^{-1} \mathbf{Y}_\nu \tag{E.5}$$

The left-handed neutrino mass matrix is diagonalized by the lepton mixing matrix U so that

$$U^T \mathbf{M}_L U = \text{diag}(m_1, m_2, m_3) \equiv \mathbf{D}_L \tag{E.6}$$

Then from Eq. (E.5) and (E.6) we will have

$$\mathbf{D}_L = U^T \mathbf{Y}_\nu^T \mathbf{M}_R^{-1} \mathbf{Y}_\nu U = U^T \mathbf{Y}_\nu^T \sqrt{\mathbf{M}_R^{-1}} \sqrt{\mathbf{M}_R^{-1}} \mathbf{Y}_\nu U \tag{E.7}$$

where the second step can be done realizing that $\mathbf{M}_R$ is diagonal. Multiplying both sides with $\sqrt{\mathbf{D}_L}$ from left and right we obtain

$$\mathbf{1} = \left[\sqrt{\mathbf{M_R^{-1}}} \mathbf{Y}_\nu U \sqrt{\mathbf{D}_L} \right]^T \left[\sqrt{\mathbf{M_R^{-1}}} \mathbf{Y}_\nu U \sqrt{\mathbf{D}_L} \right] \tag{E.8}$$

whose solution is $\sqrt{\mathbf{M_R^{-1}}} \mathbf{Y}_\nu U \sqrt{\mathbf{D}_L} = R$ for R any orthogonal matrix which can be complex provided $R^T R = 1$.

For the case of two right-handed neutrinos, R is parametrized by 1 complex angle z. Solving for R we get

$$R_{ij} = \frac{(\mathbf{Y}_\nu U)_{ij}}{\sqrt{M_i m_j}} \langle \phi^0 \rangle \tag{E.9}$$

Thus the third row of R should have the following component

$$R_{3i} = \frac{(\mathbf{Y}_\nu U)_{3i}}{\sqrt{\infty \cdot m_i}} \langle \phi^0 \rangle \tag{E.10}$$

Now for the case normal hierarchy we have $m_1 = 0$ so that R_{31} is not well defined, however since we require R to be orthogonal we can put $R_{31} = 1$. Therefore we have $R_{3i} = 0$ for $i = 2, 3$ and $R_{31} = 1$.

$$R_{NM} = \begin{pmatrix} 0 & \cos z & \pm \sin z \\ 0 & -\sin z & \pm \cos z \\ 1 & 0 & 0 \end{pmatrix} \tag{E.11}$$

In the case of inverse hierarchy we have $m_3 = 0$ therefore we should have $R_{33} = 1$ and $R_{3i} = 0$ for $i = 1, 2$

$$R_{IH} = \begin{pmatrix} \cos z & \pm\sin z & 0 \\ -\sin z & \pm\cos z & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{E.12}$$

Using the form of R from Eq. (E.11) and (E.12) we can solve for $\mathbf{Y}_\nu$ and obtain

$$\frac{v y_T^{i*}}{\sqrt{2}} = i\sqrt{m_T}\left(U_{i2}\sqrt{m_2^\nu}\cos z \pm U_{i3}\sqrt{m_3^\nu}\sin z\right) \tag{E.13}$$

$$\frac{v y_S^{i*}}{\sqrt{2}} = -i\sqrt{m_S}\left(U_{i2}\sqrt{m_2^\nu}\sin z \mp U_{i3}\sqrt{m_3^\nu}\cos z\right) \tag{E.14}$$

for normal hierarchy, and

$$\frac{v y_T^{i*}}{\sqrt{2}} = i\sqrt{m_T}\left(U_{i1}\sqrt{m_1^\nu}\cos z \pm U_{i2}\sqrt{m_2^\nu}\sin z\right) \tag{E.15}$$

$$\frac{v y_S^{i*}}{\sqrt{2}} = -i\sqrt{m_S}\left(U_{i1}\sqrt{m_1^\nu}\sin z \mp U_{i2}\sqrt{m_2^\nu}\cos z\right) \tag{E.16}$$

for inverse hierarchy.

Bibliography

[1] Strumia, A., Vissani, F., *Neutrino Masses and Mixings and ...*, arXiv:hep-ph/0606054v2, (2007)

[2] Senjanovic, G., *Course on Grand Unification*, (2007)

[3] Georgi, H., Glashow, S. L., *Unity of All Elementary Particle Forces*, Phys. Rev. Lett. 32:438-441, (1974)

[4] Bajc, B., Senjanovic, G., *Seesaw at LHC*, JHEP 0708:014, (2007)

[5] Bajc, B. Nemevsek, M., Senjanovic, G., *Probing Seesaw at LHC*, Phys. Rev. D76:055011,(2007)

[6] Abada, A. Biggio, C., Bonnet, F., Gavela, M.B., Hambye, T., *$\mu \to e\gamma$ and $\tau \to \ell\gamma$ decays in the fermion triplet seesaw model.*, JHEP 0712:061,(2007)

[7] Mohapatra, R. N., Pal, P. B., *Massive Neutrinos in Physics and Astrophysics*, Singapore, World Scientific, (2004)

[8] Peskin, M. E., Schroeder, D. V., *An Introduction to Quantum Field Theory*, Perseus Books Publishing, (1995)

[9] Weinberg, S., *Baryon- and Lepton-Nonconserving Processes*, Phys. Rev. Lett. 43:1566-1570, (1979)

[10] Cheng, T. P., Li, L. F., *Gauge Theory of Elementary Particle Physics*, Oxford, Clarendon Press, (1984)

[11] Abada, A. Biggio, C., Bonnet, F., Gavela, M.B., Hambye, T., *Low Energy Effects of Neutrino Masses*, JHEP 0712:061, (2007)

[12] Mohapatra R. N., *Unification and Supersymmetry: The Frontiers of Quark-Lepton Physics*, New York, Springer-Verlag, (2003)

[13] Ibara, A., Ross, G. G., *Neutrino Phenomenology: The Case of two right-handed neutrinos*, Phys. Lett. B 591 (2004)

[14] C. Amsler *et al.*, 2008 Review of Particle Physics, Physics Letters B667, 1, (2008)

[15] Griffiths, D. J., *Introduction to Elementary Particles Physics*, Weinheim, Willey-VCH, (2008)

[16] Lavoura, L., *General formulae for* $f_1 \to f_2\gamma$, Eur. Phys, J. C29:191-195, (2003)