HIGHER DIMENSIONAL DEFECTS IN COSMOLOGY

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ADVISOR: Prof. Jose Juan Blanco-Pillado
Dedicated to my parents,

a poor repayment for the excellent education.

And to the memory of Hans Jacobus Wospakrik,

teacher, mentor, guru.
“Ibu....durhakalah aku, jika di telapak kakimu tak kujumpai surga.”

— Fatin Hamama

“I do not know what I may appear to the world, but to myself I seem to have been only like a boy. Playing on a seashore, diverting myself in now and then finding a smoother pebble or a prettier shell than ordinary; while the great ocean of truth lay all undiscovered before me.”

— Sir Isaac Newton (1642-1727)
Abstract

Extra dimensions seem to be an important ingredient for unification of gravity with quantum field theory. Our best candidate of quantum gravity, superstring theory, requires ten-dimensional space-time for mathematical consistency. However, since our world appears four-dimensional there must be a mechanism that “hides” extra dimensions so that we do not experience them at low energy scale. There are several methods in literature for concealing extra dimensions from our naked eye. In this thesis we only focus on two of them: braneworld scenario and flux compactification, both of which require the existence of the bulk fields.

This thesis investigates the role topological defects can play as bulk fields in higher-dimensional cosmology with different asymptotic topology. The first part deals with the non-singular braneworld: Skyrme branes and its higher-dimensional generalizations. We show how these defects regularize the naked singularity around the core while at the same time approach the same flat-asymptotic behavior as the known thin-wall solutions. The second part is devoted to study an exotic transition, tunneling to (and from) nothing, in a landscape where the space-time vacua are direct products of $X_4 \times S^2$, with $X_4$ can be: anti-de Sitter ($AdS_4$), Minkowski ($M_4$), or de Sitter ($dS_4$). The tunneling is mediated by instanton solutions, via bubble nucleation. The bubble wall is smooth and magnetically-charged, and we show that this can be accomplished by having solitonic brane possessing magnetic charge, i.e., magnetic-monopole branes.
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Chapter 1

Introduction

1.1 Topological Defects

Topological defects are ubiquitous in nature. They may form whenever a phase transition occurs. They show up in many branches of physics: from everyday life to crystallography, from condensed matter to particle physics and cosmology. Their properties are so peculiar that their existence may solve some puzzles, or perhaps rule out some theories, in physics. No wonder the last four decades have witnessed an extensive amount of work on this subject [1].

So, what is topological defect?

Simply-speaking, it is a non-trivial field configuration. This configuration is made possible due to the non-trivial boundary conditions (topology) it entails, render it stable characterized by the conserved topological numbers. Examples of defects are: domain walls, cosmic strings, monopoles, and textures.

Topological defects, or equivalently topological solitons, are subgroup so-
lutions of a larger group of objects called solitons \[2, 3\], classical solutions of non-linear equations possessing finite and localized energy. In a strict mathematical language, these objects should better be called solitary waves instead, since solitons possess an additional more-restrictive property, which only a few can satisfy: that its profile should be undistorted against collisions. Physicists, however, tend not to be bothered with the distinction between the two and use the word solitons and solitary waves interchangeably. Topological solitons then are a subclass of solitons which are static and stable due to topological considerations. Solitons which do not have this topological property are called non-topological solitons \[4\], and they also play some important roles in field theories \[5\].

Once we know that some theories admit defect solutions, their existence in cosmology is almost inevitable. This is due to the so-called Kibble mechanism \[6\]. The mechanism relies on the fact that below some critical temperature, \( T_c \), the value of field \( \phi \) falls into the degenerate ground states. It then freezes-out because thermal fluctuations are no longer sufficient to lift it over the potential barrier and settle in other minimum. The choice of minimum is determined by random fluctuations and naturally expected to differ in different regions of space. The typical scale beyond which the fields \( \phi \) are uncorrelated is characterized by the correlation length \( \xi \). The magnitude of \( \xi \) depends on the detail of the corresponding phase transitions, but we can set its upper bound. Since no information travels faster than the the speed of light, the values of fields in the regions separated by scales greater than the causal horizon, \( \ell_H \), are not expected to be correlated, \( \xi \leq \ell_H \[6, 7\]. Therefore after a phase transition(s) we are left with many regions of random direc-
tions/phases of the fields. As the universe expands these regions coalesce; if two or more regions lie on the same vacuum then they can be deformed continuously into one another and we are left with a rather homogeneous distribution; otherwise there will exist boundaries between them. Since they belong to different vacua, they are not continuously deformable. The boundaries are stable topologically; they are topological defects. Different topology of the vacua produces different defects. Domain walls are produced when the corresponding symmetry is discrete; cosmic strings and monopoles when the symmetry is continuous.

In particle physics, unification has been an important topic and symmetry plays the major role. It is believed that at some high energy scale, Grand Unified Theory (GUT) scale, all fundamental interactions minus gravity are a mere low-energy manifestation of one single yet-still-unknown gauge group. Since today we do not live in such energy scale then the symmetry must have been broken down to ours. They are broken through the mechanism called *Spontaneous Symmetry Breaking* (SSB), where the vacuum is not invariant under the full symmetry of the theory. This phenomenon, combined with the Hot Big Bang scenario in cosmology, suggests that in the early universe (several) cosmological phase transitions may have happened, and as a result (a network of) topological defects may have formed. Since they are topologically stable they may survive until today or leave observable signatures in the universe, e.g., Cosmic Microwave Background (CMB), Ultra-high energy Cosmic Rays (UHECR), gravitational waves, etc. In this sense, the existence of topological defects serves as a direct link between particle physics and cosmology.

Defects coming from continuous symmetries can further be classified into
global and local. This refers to the corresponding symmetry the system possesses, global or gauge (local). Local defects have a localized profile and energy density. Outside the core the field rapidly goes to the vacuum and the energy density exponentially decreases. This is due to the existence of gauge fields, which can be gauge-transformed in such a way that the energy coming from the scalar field is compensated at large distances, so that the total energy is finite. Local (or gauged) defects are topological solitons in the real sense. This property is absent in the case of global defects. With no gauge field to compensate the energy is divergent; global string is logarithmically divergent while global monopole diverges linearly. This should not trouble us since in reality the divergence is cut off by a length scale coming from the distance between defects [8]. Another property is that since the energy is not fully localized at the core there is a tail in the energy density profile. This results in the existence of long-range force corresponding to the massless Goldstone boson associated with the breaking of global symmetry.

But symmetry-breaking defects are not the only defects we can have. In fact, the first theoretical defects were found in non-linear field theory by Skyrme [9] in 1961, before the advent of Higgs mechanism, without any appeal to phase transitions. Skyrme obtained stable classical solutions of a scalar (meson) field theory with a non-linear kinetic term in three dimensions which, upon quantization, is interpreted as baryons characterized with topological baryon numbers. This theory, later known as the Skyrme model, is a good phenomenological description of low energy QCD.
1.1.1 The Outline

This dissertation studies the role of two defects, Skyrmions and magnetic monopoles, in higher-dimensional theories, i.e., in particular their applications in braneworld cosmology and flux compactifications. In this first chapter we discuss the existence of topological defects in flat space and their stability criteria. We will begin with (perhaps) the simplest of all defects: kinks and domain walls. Then we jump into magnetic monopoles. We discuss how Dirac’s monopoles can quantize the electric charge. We follow the discussion with a non-abelian theory where magnetic monopoles appear as solitons in its spectrum. We then review the Skyrme model; from its predecessor theory: the non-linear sigma model, to Skyrmions as baryons. Finally, we will discuss the possible existence of extra dimensions and how they (if exist) can be hidden from our naked eye. In particular, we discuss several mechanisms to hide extra dimensions, e.g., Kaluza-Klein compactification and braneworld scenario. This chapter ends with the discussion of semi-classical (non-perturbative) instability of a 5d Kaluza-Klein model via nucleation of bubble with no (classical) space-time inside, a bubble of nothing.

The main body of this dissertation is divided in two parts. The first deals with Skyrmions as non-singular branes and the possibility of their higher-dimensional extensions. This is contained in chapter two. There we smooth out the naked singularity in cosmic p-branes solutions by means of a stable uncharged defect, the skyrme branes. We studied the solutions numerically and found two fundamental branches, one of which is (likely to be) unstable. The two branches of solutions merge for some critical value of gravitational
coupling constant, \( \kappa = \kappa_{\text{crit}} \), beyond which no static solutions exist. We then proceed to higher-than-three co-dimensional braneworld, where Skyrme model is no longer adequate to regularize the branes, \( i.e. \), more higher-order kinetic terms are needed. We propose an economical solution to this problem by considering non-linear sigma model in \textit{Born-Infeld} form. We show that this theory possesses natural scale in co-dimension \( d > 3 \) and so is a good candidate for generalizing the Skyrme branes in any higher dimensions. We present the solutions for several \( d > 3 \) cases.

The second part (\( i.e. \), Chapter three and four) discusses non-perturbative instabilities in flux compactifications. In the third chapter we discuss a flux compactification scenario in the framework of 6\( d \) Einstein-Maxwell theory. The compactification solutions are semi-classically unstable, \( i.e. \), they can decay to lower-energy vacua by nucleating bubbles of true vacuum. This (semi-classical) process is mediated by (gravitational) instantons interpolating between the false and true vacua. The shape of effective 4\( d \)-potential suggests two possible decay channels: (i) tunneling to smaller number(s) of flux vacua and, (ii) (de)compactification to (and from) 6\( d \) de Sitter universe. We study the third possibility: the extreme case of flux tunneling, \( i.e. \), decay to a state with no (classical) space-time, \textit{a bubble of nothing}. The bubble wall is described by a smooth magnetically-charged solitonic brane whose asymptotic flux is precisely that threading the compactified extra dimensions. This brane takes a natural form of a solitonic (‘t Hooft-Polyakov) monopole. We calculate the instantons mediating the decay and the brane solutions from three possible vacua: \( AdS_4 \times S^2 \), \( \mathbb{R}^{3,1} \times S^2 \), and \( dS_4 \times S^2 \).

Chapter four is devoted to the study of quantum cosmology of our model.
Investigating the parameter space of bubbles of nothing we found some solutions which should more appropriately be interpreted as instantons from nothing. These are solutions of Euclidean field equations which represent the spontaneous creation of an open universe out of nothing via (semiclassical) quantum tunneling. One important significance of this solution is that the instanton is non-singular. Upon analytic continuation to Lorentzian spacetime, our solutions describe the evolution of bubbles from nothing right after nucleating. Interestingly, there exist some regions in parameter space whose values dictate the inflationary-type evolution of the 4d universe while leaving the extra dimensions compactified. We are able to obtain up to $\sim 17$ e-folds. Any realistic theory should ensure the existence of a period of slow-roll inflation, that is by having no less than 60 e-folds. Nevertheless, our model can be a good toy-model description of creation of our (open) universe and its subsequent evolution.

Finally, we will give an overall summary and conclusion of this dissertation in the fifth Chapter.

1.1.2 The Simplest Defects: Kinks and Domain Walls

To illustrate the properties of topological defects it is instructive to discuss (possibly) the simplest topological defect found in a 1 + 1-dimensional real scalar field theory with a double-well potential, a kink \cite{2}. Consider a Lagrangian density

$$
\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{\lambda}{4} (\phi^2 - \eta^2)^2.
$$

(1.1)
This theory has a $\mathbb{Z}_2$-symmetry, i.e., the Lagrangian is invariant under $\phi \rightarrow -\phi$. The minima of the potential are degenerate, $\phi = \pm \eta$. Apart from those constant solutions, the system also possesses non-trivial static solutions interpolating between those vacua, and we will see that this configuration is stable, characterized by a non-zero integer number.

The Euler-Lagrange equation gives the second-order equation of motion. There is a smart method by Bogomolny [10] that reduces the problem to first-order and obtains exact solutions. For a static system, $\phi = \phi(x)$, the total energy is

$$E = \int dx \left( \frac{1}{2} \phi'^2 + V(\phi) \right),$$

(1.2)

where prime denotes derivative with respect to $x$, and $V(\phi)$ refers to the double-well potential term. Bogomolny noticed that the energy can be rearranged by completing the square as follows

$$E = \int dx \frac{1}{2} \left( \phi' \mp \sqrt{2V(\phi)} \right)^2 \pm \int d\phi \sqrt{2V(\phi)}.$$  

(1.3)

It is clear that there is an absolute lower bound of energy given by $\| \int d\phi \sqrt{2V(\phi)} \|$, and that the energy is minimized when the bound is saturated. This happens if and only if the first term is zero, i.e.,

$$\phi' = \sqrt{2V(\phi)}.$$  

(1.4)

Solving the equation of motion and using our form of potential, we obtain the exact solutions, the “kink”,

$$\phi = \eta \tanh \left( \sqrt{\frac{\lambda}{2}} \eta x \right).$$

(1.5)

These solutions are non-dissipative, finite-energy, solutions. They are time-independent (static) solutions which can be Lorentz-transformed (boosted) to
any arbitrary velocities.

We can easily see that this solution is stable since it interpolates between two distinct vacua at \( \pm \infty \) so deforming it to vacuum solution requires an infinite amount of energy. Another way of seeing this is to acknowledge the existence of a current,

\[
j^\mu = \epsilon^{\mu\nu} \partial_\nu \phi, \tag{1.6}\]

which is conserved trivially, \( \partial_\mu j^\mu = 0 \), irrespective of any equation of motion the field satisfies, and gives rise to a conserved charge

\[
Q = \int dx j^0 = \phi(+\infty) - \phi(-\infty), \tag{1.7}\]

which is non-zero due to the boundary conditions imposed. For this reason they are called topological current and topological charge, respectively. Thus the solutions are characterized by the topological number where different solutions corresponds to different charge, and since each belongs to different topology then they cannot be deformed into one another continuously.

Another striking property of topological defects is that they are non-perturbative objects. To see this, define a mass scale \( m \equiv \sqrt{\lambda \eta} \), coming from the perturbation of the vacuum. The total energy can be written

\[
E = \int d\phi \sqrt{2V(\phi)},
\]

\[
= \frac{2\sqrt{2}}{3} \sqrt{\lambda \eta^3},
\]

\[
= \frac{2\sqrt{2}}{3} \frac{m^3}{\lambda}, \tag{1.8}\]

the coupling constant \( \lambda \) is inversely proportional to the energy (or the rest

\[1\] The discussion here follows that of [1].
mass) of the defect! The smaller the coupling is, the heavier the defect will be. This means that the properties of these exact solutions cannot be revealed by perturbation method; i.e., they will not appear on any order of perturbation theory.

We can extend the system to three spatial dimensions trivially, by keeping the dependence only on one of them. What we obtain now is a 2-dimensional sheet-like wall separating out two different vacua in one direction (say, $x$) and extending out to infinity in two other directions, a domain wall. Due to the topological charge it endows once they form they will be stable indefinitely.\footnote{If the universe in its early history possessed $\mathbb{Z}_2$-symmetry and underwent symmetry breaking then domain walls may have formed and they may survive until today. Clearly this will have a significant impact on cosmological signatures. In fact, Zeldovich et al.\cite{7} shows that theories with domain walls in their spectrums must be ruled out since they will over-close the universe and contradict the observations.}

\section{1.2 Topology of the Vacuum Manifold}

As argued above, the topology of the vacuum manifold is essential in determining the existence (and types) of the defects formed. We will classify defect solutions based on their corresponding vacuum manifolds. The more detail discussion on basic topology and homotopy theory are discussed in Appendix A.\footnote{The discussion here will largely follow \cite{1,12,13}.}

Using the powerful fundamental theorems of homotopy theory we can classify defects solutions based on their corresponding vacuum manifolds. Let $G$ be the group of the corresponding field theory and let $H$ be a subgroup of $G$, $H \subset G$, under which the vacuum manifold remains invariant. Domain walls occur if the vacuum manifold is disconnected, i.e., when the zeroth homotopy
group is non-trivial; e.g., $G = \mathbb{Z}_2$, $H = \mathbb{I}$, so $\pi_0(G/H) \cong \pi_0(\mathbb{Z}_2) = \mathbb{Z}_2$. Strings occur if the vacuum manifold is non-simply connected, i.e., the first homotopy group is non-trivial; e.g., $G = U(1) \cong S^1$, $H = \mathbb{I}$, so that $\pi_1(G/H) \cong \pi_1(S^1) = \mathbb{Z}$. Monopoles occur when the vacuum is non-contractible, i.e., the second homotopy group is non-trivial; e.g., $G = SU(2)$, $H = U(1)$, so that $\pi_2(SU(2)/U(1)) \cong \pi_1(U(1)) \cong \pi_1(S^1) = \mathbb{Z}$, by virtue of the second fundamental theorem. Lastly, defects called textures occur when the third homotopy group is non-trivial; e.g., $G = SU(2)$, $H = \mathbb{I}$, so that $G/H = SU(2)/\mathbb{I} \cong S^3$, and $\pi_3(S^3) = \mathbb{Z}$.

### 1.3 Derrick’s Theorem

Defects classified above based on the homotopy classes may or may not be stable in static condition. This is due to energetic consideration; whether the energy is stable against all variations including spatial rescaling, i.e., that the energy is minimum.

There exists an elegant theorem by Derrick [14] (and independently by Hobart [15]) that states that in most (pure-scalar-field) field theory there do not exist any stable-finite-energy static solutions, except the vacuum; i.e., no topological solitons exist. This theorem is essentially scaling argument: we perturb the size of the defects and see whether the energy is in the minima or

---

4We might be tempted to think that since $H$ is trivial then $\pi_0(H) = \mathbb{I}$ and therefore string-defects do not exist. But this is incorrect, since $U(1) \cong S^1 \cong \mathbb{R}^1/\mathbb{Z}$, and since $\mathbb{R}^1$ is a connected and simply-connected continuous group, then $\pi_1(S^1) \cong \pi_0(\mathbb{Z}) = \mathbb{Z}$, by the first fundamental theorem.

5Another way of seeing this is that since $SU(2)/U(1) \cong S^2$, then $\pi_2(S^2) = \mathbb{Z}$.

6Later we will see that the Skyrme model is a generalization of textures. Both Skyrmions and textures belong to a class of scalar-field theory called non-linear sigma model.

7Notice that global defects evade this argument by having divergent energy density.

8See Appendix B for a more comprehensive discussion.
not. If it is, then the defects will have a definite size and the perturbation is stable. Otherwise, the defects will be vulnerable to expansion or collapse. Appendix B shows that magnetic monopoles and Skyrme model possess natural scale, thus stable.

1.4 Magnetic Monopoles

1.4.1 Dirac Monopoles

Magnetic monopoles are one of the “holy grails” in physics. They are predicted—or rather expected—by symmetry, yet its experimental detection is still elusive and subject to experimental search.

Consider the famous Maxwell’s equations (in covariant form):

$$\partial_\nu F^{\mu \nu} = j^\mu,$$

$$\partial_\mu \tilde{F}^{\mu \nu} = 0,$$

(1.9)

where $\tilde{F}^{\mu \nu} \equiv \frac{1}{2} \epsilon^{\mu \nu \alpha \beta} F_{\alpha \beta}$, the dual field strength. They will enjoy a perfect duality symmetry

$$F^{\mu \nu} \rightarrow \tilde{F}^{\mu \nu},$$

$$\tilde{F}^{\mu \nu} \rightarrow -F^{\mu \nu},$$

(1.10)

if there exists a magnetic current, $j^\mu_m$, such that the second equation satisfies

$$\partial_\mu \tilde{F}^{\mu \nu} = j^\nu_m.$$ The non-existence of magnetic monopoles thus spoils this perfect electric-magnetic duality.

---

9The others are, among them, Higgs particles and quantum gravity.

10Obviously Bianchi identity no longer holds in this case.
In 1931 Dirac [16] showed that, by invoking quantum mechanics, the existence of monopoles will necessarily make the electric charge quantized. He derived the following relation:

\[ 2\pi n = \frac{e}{\hbar c} 4\pi g, \]  

(1.11)

or

\[ eg = \frac{n\hbar c}{2}, \]  

(1.12)

the famous Dirac quantization condition. This equation tells us that if there exists a magnetic monopole somewhere in the universe then the electric charge must necessarily be quantized. Conversely, the monopole charge is also quantized in unit of inverse electric charge. Since in nature electric charge is indeed quantized (in unit of electron charge) and so far no other (if no at all) compelling theory that can explain why, no wonder the theoretical idea of magnetic monopoles is very attractive.

### 1.4.2 ’t Hooft-Polyakov Monopoles

The quantization condition (1.12) implies the necessity of electric charge quantization if any magnetic monopole exist, but does not say that it should exist. Maxwell’s equations are indeed more symmetric with magnetic monopoles, but there is no necessity why it has to be. The existence of monopoles is postulated in the first place; i.e., in some sense, ad hoc. In this section we will discuss a class of theory where monopoles must necessarily exist; its existence is required for mathematical consistency.

Dirac monopoles are based on abelian (Maxwell) gauge theory, with corresponding symmetry \( U(1) \). They are singular at the origin, and their total
energies diverge linearly. The monopole charge is quantized, characterized by integer numbers $n$. If we up-grade the symmetry to a higher group the corresponding gauge theory then is non-abelian, \textit{i.e.}, \textit{Yang-Mills}. In 1974, 't Hooft \cite{17} and Polyakov \cite{18} independently discovered that some non-abelian field theories admit magnetic monopoles as their static solutions.

The simplest corresponding group for non-abelian gauge theory is $SU(2)$. Alternatively we can have gauge group $SO(3)$ with Higgs fields $\phi^a$, $a = 1, 2, 3$, transforming in a fundamental triplet representation. In the context of particle physics, the gauge group $SO(3)$ was used by Georgi and Glashow \cite{19} to construct electroweak unification. It was once a competitor of the $SU(2) \times U(1)$ Weinberg-Salam model \cite{20}. The discovery of weak neutral-current processes mediated by $Z$ boson in 1974 at CERN favored the latter as the established electroweak theory and rendered the Georgy-Glashow model obsolete. Unfortunately, Weinberg-Salam model does not have magnetic monopoles in its spectrum\cite{11} due to the non-compactness of its covering group. But if we believe in the unification of all subnuclear forces, \textit{i.e.}, Grand Unified Theory\cite{12}, monopoles should exist generically in all grand unification models since they have an unbroken compact covering group, $U(1)$, mediating electromagnetism. The monopole will then have energy around GUT scale.

For the sake of concreteness, consider the Yang-Mills-Higgs Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} - \frac{1}{2} D_\mu \phi^a D^\mu \phi^a - V(\phi),$$

with $V(\phi) = \frac{\lambda}{4} (\phi^a \phi^a - \eta^2)^2$, chosen to develop vacuum expectation value at $\phi^a \phi^a = \eta^2$. This VEV breaks the symmetry spontaneously, $SO(3) \rightarrow SO(2)$,

\textsuperscript{11}It does, however, possess an unstable-saddle-point solution, a \textit{sphaleron} \cite{21}.

\textsuperscript{12}For example, $SU(3) \times SU(3)$-Weinberg model \cite{22} or $SU(5)$ Georgi-Glashow model \cite{23}.
making the vacuum manifold degenerate into $\mathcal{M} = SO(3)/SO(2) \cong S^2$. The relevant homotopy group is $\pi_2(S^2) = \mathbb{Z}$; the vacuum is non-contractible, characterized by integer winding number $k \in \mathbb{Z}$.

Apart from constant solutions $\eta$, $\phi^a$ can have a non-trivial configuration. 't Hooft [17] and Polyakov [18] proposed a spherically-symmetric ansatz:

$$\phi^a = \eta p(r) \frac{x^a}{r}, \quad \text{(1.14)}$$

where $p(r)$ is a function of radial coordinate with boundary conditions $p(0) = 0$, $p(\infty) \to 1$, and $x^a$ are three-dimensional spatial coordinates. This form looks peculiar as it mixes coordinate- and field-spaces. For this reason, Polyakov called it *hedgehog* ansatz [18]. To compensate the gradient of $\phi^a$ at infinity, the gauge field should take the following form

$$A_0^a = 0, \quad A_i^a = -(1 - w(r)) \epsilon^{aij} \frac{x^j}{er^2}, \quad \text{(1.15)}$$

with $w(r)$ behaving like $w(0) = 1$, $w(\infty) \to 0$. In general there exists no known analytical solution for the corresponding Euler-Lagrange equations. However, using the Bogomolny trick exact solutions for vanishing-potential ($\lambda = 0$) case are known. We discuss it in Appendix [C].

The boundaries at the origin make both fields regular, while at infinity ensure that the energy density is finite. These asymptotic condition, in turn, forces the field strength tensors to be

$$\mathcal{F}_{0i}^a = 0, \quad \mathcal{F}_{ij}^a \to \epsilon_{ijk} \frac{x^a x^k}{er^3}. \quad \text{(1.16)}$$
The first equation states that there is no (Yang-Mills) “electric” fields present. But the second equation seems to imply that there exists “magnetic” fields which goes like Coulomb force, \( \sim \frac{1}{r^2} \). This is counter-intuitive since in abelian Maxwell’s theory magnetic field \( \mathbf{B} \) is driven by electric current \( \mathbf{j} \).

It is true that most of the fields in (1.16) are non-physical. We cannot interpret \( A^3_\mu \), say, as electromagnetic potential in Maxwell’s theory. This is because the Higgs field \( \phi^a \) does not satisfy unitary gauge asymptotically, \( \phi^a(\infty) \to \delta^{a3} \). However, we can make the correspondence in the previous paragraph more exact by defining, following ’t Hooft’s proposal [17], a gauge-invariant quantity

\[
\mathcal{F}_{\mu\nu} \equiv \frac{\phi^a}{|\phi|} \mathcal{F}^a_{\mu\nu} - \frac{1}{e|\phi|^3} e^{abc} \phi^b (D\phi^c) (D\phi^c).
\]

(1.17)

It is trivial to check that for unitary gauge it reduces to

\[
\mathcal{F}^3_{\mu\nu} = \partial_\mu A^3_\nu - \partial_\nu A^3_\mu,
\]

(1.18)

naturally interpreted as the Maxwell’s field strength tensor. Using our hedgehog ansatz (1.14) and eq.(1.16) its asymptotic form is given by

\[
\mathcal{F}_{0i} = 0,
\]

\[
\mathcal{F}_{ij} \to \epsilon_{ijk} \frac{x^k}{e r^3}.
\]

(1.19)

Now we can legitimately interpret these results as the real electric and magnetic fields at infinity,

\[
\mathbf{E} = 0,
\]

\[
\mathbf{B} = \frac{1}{e r^3} \mathbf{r}.
\]

(1.20)

Since no electric current drives this magnetic field, we are forced to accept that it is sourced by a magnetically-charged particle; that there exists a magnetic
monopole with magnetic charge, \( g \), given by

\[
g = \frac{1}{e},
\]

(1.21)

appearing naturally as a result of boundary conditions imposed at infinity for the energy to be finite. This charge satisfies (Schwinger’s \cite{24}) quantization condition\(^{13}\)

\[
eg g = 1,
\]

(1.22)

in units where \( \hbar = c = 1 \).

t’Hooft \cite{17} gives some estimate of the mass of the monopole\(^{14}\), \( m \), and finds that it is given by

\[
m = \frac{4\pi}{e^2}m_v f(\lambda/e^2),
\]

(1.23)

where \( m_v \sim e\eta \) is the mass of the vector particles and \( f(x) \) is a monotonically-increasing function of \( x \). Variational methods (and later verified by numerical calculation) show that the function is quite slowly-varying, satisfying \( f(0) = 1 \) and \( f(\infty) = 1.787 \), does not depend much on \( \lambda \). This suggests that the mass is quite independent of \( \lambda \) as well.

We can see that monopoles are classical objects from the following argument. The size of its core, \( R_c \), is determined by by the width of the profile functions \( p(r) \) and \( w(r) \), \( r_H \sim m_H^{-1} \) and \( r_v \sim m_v^{-1} \) respectively, before they relax to the vacuum. Assuming they are of comparable scale, \( r_H \sim r_v \), then \( R_c \sim r_H \sim r_v \). The Compton wavelength of the monopole is

\[
\lambda_{\text{Compton}} \sim m^{-1},
\]

(1.24)

\(^{13}\)It is twice the Dirac condition (1.12).

\(^{14}\)Which can be done using variational arguments. See also \cite{25, 1, 13}.
the wavelength is smaller than the core size by a factor of $e^2/4\pi$, $\lambda_{\text{Compton}} \ll R_c$. Thus it is safe to consider monopoles as classical objects.

### 1.5 Skyrme Model

The Skyrme model \cite{9, 26} is a non-linear three-dimensional theory of pions. Originally proposed as a unified theory of strong interactions, it is now an effective theory of low-energy QCD \cite{27, 28}. The model is a subclass of a non-linear field theory called non-linear sigma model \cite{27, 28} (or textures, in cosmology). However, unlike textures which are unstable to collapse in $3 + 1$ dimensions, Skyrme model has topological soliton solutions interpreted as baryons \cite{15}, called Skyrmions.

#### 1.5.1 Non-linear Sigma model

In field theory, a field configuration is a mapping from a manifold $\mathcal{N}$ (representing space-time endowed with space-time metric $g_{\mu\nu}(x)$) to a target manifold $\mathcal{M}$, the field space \cite{29}. In most cases, manifold $\mathcal{M}$ is flat. In non-linear sigma model $\mathcal{M}$ is a non-flat Riemannian manifold (endowed with internal metric $G_{ab}(\phi)$), isomorphic to a coset space $G/H$. Initially it was proposed by Gell-Mann and Levy \cite{30} in the context of $\beta$-decay theory. Later it gained wide acceptance, in particular in the study of spontaneous compactifications \cite{29, 31, 32, 33, 34}. The origin of the name of this theory comes from the fact that we can model the three-dimensional (non-linear) pion

\footnote{Characterized by topological baryon number $B$.}
theory with
\[ \Phi^i = (\sigma, \vec{\pi}), \quad (1.25) \]
where \( \vec{\pi} \) is the triplet of pion fields and \( \sigma \) is an auxiliary field determined through a non-linear constraint \[ \sigma^2 + \vec{\pi} \cdot \vec{\pi} = 1. \quad (1.26) \]

In more than two dimensions this theory is non-renormalizable. For \( d = 2 \) its renormalizability is proved by Friedan \[35\].

The Langrangian of this model is given by
\[ \mathcal{L} = \frac{1}{2} G_{ab}(\phi) g^{\mu\nu}(x) \partial_{\mu} \phi^a \partial_{\nu} \phi^b, \quad (1.27) \]
where \( G_{ab}(\phi) \) is the internal metric on the manifold \( \mathcal{M} \), \( g_{\mu\nu} \) is the space-time metric, and \( \phi^a, a = 1, 2, ..., n, \) are the internal coordinates of the field space \( \mathcal{M} \). This Langrangian takes a familiar form in a well-known theory in a branch of differential geometry, the theory of Harmonic Maps \[36, 37, 38\]. By varying eq.\((1.27)\) with respect to \( \phi^a \) we obtain the equations of motion
\[ \frac{1}{\sqrt{g}} \partial_\mu \left( \sqrt{g} g^{\mu\nu} \partial_\nu \phi^a \right) + g^{\mu\nu} \Gamma^a_{\mu b c} \partial_\mu \phi^b \partial_\nu \phi^c = 0, \quad (1.28) \]
where \( g \) is the determinant of the space-time metric, and \( \Gamma^a_{\mu b c} \) are the Levi-Civita connections of the target manifold,
\[ \Gamma^a_{\mu b c} = \frac{1}{2} G^{ad} \left( \frac{\partial G_{db}}{\partial \phi^c} + \frac{\partial G_{dc}}{\partial \phi^b} - \frac{\partial G_{bc}}{\partial \phi^d} \right). \quad (1.29) \]

Topological defects coming from non-linear sigma model are called textures \[10\]. Consider a general scalar field theory with Langrangian

\[ \mathcal{L} = \frac{1}{2} G_{ab}(\phi) g^{\mu\nu}(x) \partial_{\mu} \phi^a \partial_{\nu} \phi^b, \quad (1.27) \]

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Topological defects coming from non-linear sigma model are called textures \[10\]. Consider a general scalar field theory with Langrangian

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given by

\[ \mathcal{L} = \frac{1}{2} \partial_\mu \Phi^i \partial^\mu \Phi^i - V(\Phi), \tag{1.30} \]

where \( i = 1, 2, \ldots, N \). The potential is assumed to be such that its corresponding symmetry \( G = SO(N) \) is spontaneously broken, \( V(\Phi) = \frac{\lambda}{4}(\Phi^i \Phi^i - \eta^2)^2 \). As a result the vacuum manifold \( \mathcal{M} \) respects only \( H \) group symmetry \( (H \subset G) \).

As an example, consider the case when a global \( SU(2) \) is broken completely to \( I \), or, equivalently, when an \( SO(4) \) broken down to \( SO(3) \). The third homotopy group \( \pi_3 \) is non-trivial, \( \pi_3(\mathcal{M}) \neq I \).

Texture defects exist due to the non-triviality of the third homotopy group, and they are characterized by (topological) integer numbers. As long as their size is much larger than the symmetry-breaking scale \( \ell_{SB} \sim \sqrt{\lambda} \eta \), the massive modes are effectively frozen and so \( \Phi^i \) are constrained to everywhere stay in \( \mathcal{M} \), \( \Phi^i \Phi^i = \eta^2 \). The fields \( \Phi^i \) then can be parametrized by “angles” \( \phi^a \),

\[ \Phi^i = \Phi^i(\phi^a). \tag{1.31} \]

In this regard, the low-energy Lagrangian is effectively given by eq.(1.27), with

\[ G_{ab} = \frac{\partial \Phi^i}{\partial \phi^a} \frac{\partial \Phi^i}{\partial \phi^b}, \tag{1.32} \]

parametrizing the metric on manifold \( \mathcal{M} \). The vacuum manifold is topologically a three-sphere, \( \mathcal{M} = SO(4)/SO(3) \cong S^3 \). The non-triviality are characterized by winding number derived from topological current density \[ \sqrt{-g} j^\mu = \frac{1}{12\pi^2 \eta^4} \epsilon^{\mu \nu \alpha \beta} \epsilon_{abcd} \Phi^a \partial_\nu \Phi^b \partial_\alpha \Phi^c \partial_\beta \Phi^d. \tag{1.33} \]

---

\(^{18}\)The discussion here follows refs. \[1\] \[40\].

\(^{19}\)That is, at low enough energy.
As already discussed, topology does not guarantee the existence of stable solutions. By Derrick’s theorem we can see that a three-dimensional texture is unstable to collapse\textsuperscript{20}.

Similarly in higher dimensions, “extended-textures” can arise in non-linear scalar field theories possessing (global) $SO(N)$ symmetry broken spontaneously to $SO(N-1)$ and imposing spatial uniformity at infinity. The vacuum manifold will topologically be a $(N-1)$-sphere, $\mathcal{M} = SO(N)/SO(N-1) \cong S^{N-1}$.

### 1.5.2 Skyrmions as Topological Solitons

In the beginning of 1960’s, Tony Hilton Royle Skyrme proposed a theory of strong interactions based on pion fields alone\textsuperscript{9, 26}. Contrary to Yukawa theory which puts fermions as fundamental fields, he built a unification of mesons and baryons with pions as fundamental fields. Nucleons are later obtained as certain classical configuration of the pion field\textsuperscript{43}. He achieved it by generalizing the non-linear sigma model. The insufficiency of Lagrangian (1.27) to produce stable defects is cured by introducing an additional kinetic term fourth-order in its field derivatives, the Skyrme term. This term scales differently with the ordinary kinetic term under spatial rescaling and as a result the defects have natural scale. Skyrme further interpreted these solutions, the skyrmions, as baryons with baryon numbers coming from topological winding number.

It is often more convenient to study the model by defining the Skyrme field\textsuperscript{21} $U(t, x)$, as a unitary $2 \times 2$-scalar matrix transforming under $SU(2)$.

\textsuperscript{20}However, the story will be different if we couple it to gravity. The topology of the space-time allows one to find stable maps from the latter to the space of the fields\textsuperscript{29}.

\textsuperscript{21}Also called chiral field.
To satisfy finiteness of energy the chiral fields must be constant at infinity. Without loss of generality we can take it to be unity

$$\lim_{r \to \infty} U(t, r) \to 1.$$ (1.34)

This boundary conditions effectively defines a one-point compactification of $\mathbb{R}^3$, i.e., $\mathbb{R}^3 + \infty \cong S^3$, so that topologically

$$U : S^3 \to S^3.$$ (1.35)

We know that its third homotopy group is non-trivial, $\pi_3(S^3) = \mathbb{Z}$.

The Skyrme Lagrangian is given by

$$\mathcal{L} = \frac{F_\pi^2}{16} Tr(L_\mu L^\mu) + \frac{1}{32e^2} Tr ([L_\mu, L_\nu][L^\mu, L^\nu]),$$ (1.36)

with $L_\mu$ is called left-chiral current, given by $L_\mu \equiv U^\dagger \partial_\mu U$ and trivially satisfying the Maurer-Cartan identity

$$\partial_\mu L_\nu - \partial_\nu L_\mu + [L_\mu, L_\nu] = 0,$$ (1.37)

$F_\pi \simeq 189$ MeV is pion decay constant, $e$ is the dimensionless Skyrme constant. The importance of eq.(1.36) compared to the non-linear sigma model is on the second term, the Skyrme term. It is the non-linear term that stabilizes the solutions. Clearly any other higher order term in derivatives will do as well. But the uniqueness of Skyrme term lies on the fact that it is the only fourth order Lorentz-invariant term which does not introduce higher terms than second order time derivatives in the equations of motion. Thus this term will not produce any ghost upon quantization.
From the variational principle we obtain the field equations

\[ \partial_{\mu} \left( L^{\mu} - \frac{1}{4e^{2}F_{2}} [L_{\nu}, [L^{\mu}, L^{\nu}]] \right) = 0. \quad (1.38) \]

Skyrme proposed a hedgehog ansatz\(^{22}\)

\[ U = \exp \left( i f(r) \hat{r} \cdot \vec{\tau} \right), \]
\[ = \cos f(r) + i \vec{\tau} \cdot \hat{r} \sin f(r), \quad (1.39) \]

where \( f(r) \) is the Skyrmion’s profile function (also known as chiral angle) and \( \vec{\tau} \) are the \( 2 \times 2 \) Pauli matrices. Notice that we can identify the chiral field as the sigma model representation

\[ U = \sigma + i \vec{\tau} \cdot \hat{\pi}. \quad (1.40) \]

It is easy to see that in this representation the unitarity of \( U \) implies (1.26).

For static case, the field equation reads

\[ \left( 1 + \frac{2}{r^2} \sin^2 f \right) \frac{\partial^2 f}{\partial r^2} + \frac{2 \partial f}{r \partial r} + \left( \frac{\partial f}{\partial r} \right)^2 \left( \frac{\sin 2f}{r^2} \right) - \frac{\sin 2f}{r^2} \left( 1 + \frac{\sin^2 f}{r^2} \right) = 0. \quad (1.41) \]

This is a non-linear second-order differential equation with no known analytic solution. However, as in the case of monopoles it is straightforward to find numerical solution by means of shooting method. We need to impose the following boundary conditions

\[ f(0) = n \pi, \quad f(\infty) = 0. \quad (1.42) \]

Numerical calculation reveals the shape of the profile function; it is a monotonically-decreasing function interpolating between the boundary conditions.

\(^{22}\)It corresponds to soliton with topological number \( n = 1 \).
Being topologically stable, Skyrmion is characterized by topological baryon number

\[ B = -\frac{1}{24\pi^2} \int d^3x \varepsilon^{abc} Tr(L_a L_b L_c), \]  

which is equivalent to the charge coming from (1.33). Using Bogomolny analysis it can be shown that this topological charge provides the lower bound for the energy, as follows

\[ E = -\int d^3x Tr \left( \frac{F^2}{4} L_i L^i + \frac{1}{16e^2} (\varepsilon_{ijk} L_j L_k)(\varepsilon^{ilm} L^l L^m) \right). \]  

(1.44)

With trivial algebra we can re-arrange it as follows

\[ E = -\int d^3x Tr \left[ \left( \frac{F^2}{4} L_i - \frac{1}{4} \varepsilon_{ijk} L_j L_k \right)^2 - \frac{F^2}{16e} \varepsilon_{ijk} L_i L_j L_k \right]. \]  

(1.45)

The first term is positive-definite. This implies the boundedness of energy from below

\[ E \geq \int d^3x \frac{F^2}{4e} Tr |\varepsilon_{ijk} L_i L_j L_k|. \]  

(1.46)

The topological current can be defined as

\[ B^\mu = -\frac{1}{24\pi^2} Tr (\varepsilon^{\mu\nu\alpha\beta} L_\nu L_\alpha L_\beta), \]  

(1.47)

which gives precisely (1.43) for the charge. The lower bound of energy then can be expressed as

\[ E \geq \frac{6\pi^2 F^2}{e} |B|. \]  

(1.48)

Unfortunately this bound cannot be saturated. The energy of Skyrmions will always be greater than the lowest energy possible. This is due to the incompatibility between the domain and target manifolds; \textit{i.e.}, the Skyrme model is defined as a theory on \( \mathbb{R}^3 \) while the field space takes a shape of \( S^3 \), and no isometry\(^{23}\) between \( \mathbb{R}^3 \) and \( S^3 \) \(^{39}\). We cannot wrap \( \mathbb{R}^3 \) on \( S^3 \) smoothly.

\(^{23}\) The bound can, however, be saturated in a theory of Skyrme model defined \textit{on} 3-sphere, as in \(^{44}\).
The hedgehog configuration (1.14) has baryon number $B = 1$,

$$B = -\frac{2}{\pi} \int_{0}^{\infty} dr f \frac{df}{dr} \sin^2 f,$$

$$= -\frac{1}{\pi} [f - \sin 2f]_0^\pi,$$

$$= 1. \quad (1.49)$$

Apparently for $B > 1$ the hedgehog configuration is unstable. Manton [45] shows that the $B = 2$ configuration is axially-symmetric. This is confirmed by Braaten and Carson [46] who obtained $B = 2$-skyrmions in an axially-symmetric ansatz. This multi-skyrmion solution shows bound-state configuration and is interpreted as deuteron.

### 1.6 Extra Dimensional Physics

#### 1.6.1 Kaluza-Klein Theory

The idea that our world might possess more than three spatial dimensions is both old and appealing. The primary reason for considering the existence of extra dimensions is unification. In 1921, just six years after Einstein published his famous geometric theory of gravitation, Theodor Kaluza [47] proposed the first unification in modern physics, a unification of gravity with electromagnetic. He achieved it in the framework of General Relativity, by postulating (one) extra dimension in the Einstein’s equations, $x^M = (x^\mu, x^5)$. Kaluza’s idea was to consider 5d vacuum Einstein theory that, when viewed from 4d perspective, contains 4d gravity plus electromagnetic field. The ansatz of the higher-dimensional metric, $g_{AB}$ is as follows. The $\mu\nu$-part are identified with

\footnote{For a nice review on Kaluza-Klein theory from general relativistic point of view, see [48].}
the four-dimensional metric $g_{\mu\nu}$; the 4$\mu$-part are interpreted as the Maxwell fields, $A_\mu$; and a scalar 44-component, $\phi$, which, upon compactification, characterizes the size (radius) of the extra dimension, called radion.

This theory faces an immediate challenge: if extra dimension ever exists at all, why can we never observe it? This is a legitimate question since, not only that the existence of extra dimension is unimaginable and non-visualizable from our everyday reality, but also the known laws of physics (e.g., gravitational force) appear to be three-dimensional. To answer this question, Kaluza postulated the cylinder condition; the condition where all physical laws only depends on $x^\mu$ but not on $x^5$, i.e., $\partial_5 g_{AB} = 0$. The theory is described by 5$\!\!\!d$ Einstein-Hilbert Action

$$S = \frac{1}{16\pi G} \int d^5x \sqrt{-g^{(5)}} R^{(5)},$$  \hspace{1cm} (1.50)

with $A, B = 0, 1, 2, 3, 4$, $R^{(5)}$ and $\hat{G}$ are the 5$\!\!\!d$ Ricci scalar and gravitational coupling constant, respectively. Integrating out the extra dimension yields the Einstein-Maxwell Action in 4$\!\!\!d$

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g^{(4)}} \left( R^{(4)} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right);$$  \hspace{1cm} (1.51)

4$\!\!\!d$ electromagnetic fields come out of 5$\!\!\!d$ vacuum geometry. This proposal is theoretically attractive since it opens up the paradigm of unification of fundamental forces in terms of geometry.

Kaluza’s cylinder condition seems to be unnatural. Klein [50] relaxed this assumption by invoking the idea of compactification. He treated all components of $x^M$ on equal footing, but postulated that the $x^4$-coordinate does not

\textsuperscript{25}This procedure is called dimensional reduction.

\textsuperscript{26}This is valid only when $\phi = \text{constant}$. Otherwise the 4$\!\!\!d$ Action would be of Brans-Dicke type, see [48, 49].
extend to $\pm \infty$, but instead curls up with the end points identified, a circle

\[ -\infty \leq x^\mu \leq \infty, \]

\[ 0 \leq x^5 \leq 2\pi R, \quad (1.52) \]

with periodic condition imposed, $x^5(0) = x^5(2\pi R)$, and $R$ the size of the extra dimension. Klein further postulated that the size of extra dimension was small, naturally set to be the Planck scale; so small that it was unobservable at low energies. With these assumptions it can be shown that at low enough energy physics looks four-dimensional, while the effects of extra dimension can only be probed by very-high-energy particles. Let us consider a massless scalar field $\Phi$ living in 5$d$ world with the fourth-dimension compactified on a circle with radius $R$. We can Fourier-expand it as

\[ \Phi(x^M) = \sum_{n=-\infty}^{\infty} \phi_n(x^\mu)e^{inx^5/R}, \quad (1.53) \]

with $n = 0, \pm 1, \pm 2, \ldots$ indicating the periodicity of $x^5$. Massless scalar field obeys massless Klein-Gordon equation,

\[ \square_{(5)} \Phi = 0, \quad (1.54) \]

so each mode obeys

\[ \square_{(4)} \phi_n - \frac{n^2}{R^2} = 0. \quad (1.55) \]

We see that only particles with energy of order at least $\sim 1/R$ can probe the extra dimension. Particles with $E \ll 1/R$ are effectively “trapped” in our

\[ ^{27} \text{From gauge theory point of view this is } U(1)-\text{type compactification, which is consistent with the gauge field produced: Maxwell (abelian) field.} \]

\[ ^{28} \text{We only consider massless scalar field for simplicity. Other type of fields (e.g., spinor, gauge, tensor) and condition (e.g., massive, interacting, etc) can be substituted instead without having any fundamental alteration in the conclusion.} \]
From 4d point of view, Kaluza-Klein (KK) modes appear to be “particles” with mass \( m = n/R \) [51]. If, following Klein’s assumption, \( R \sim \ell_{\text{Planck}} \), then it requires \( E \sim M_{\text{Planck}} \) to open up the extra dimension. The existence of extra dimension, if any, is safely hidden from everyday-life observations.

### 1.6.2 Braneworld Picture

The Kaluza-Klein compactification effectively “hides” the extra dimension and makes physics appear four dimensional at low energies. But the assumption that the size of extra dimension is as small as the Planck scale casts a doubt whether it can ever be proved or not since such a huge energy to probe the scale is not available on earth, and will not be so in the foreseeable future. Another alternative to “conceal” extra dimension, yet more accessible to falsification, is the so-called braneworld scenario. This concept is inspired by the progress in the study of non-perturbative objects, called \( D\)-branes [52, 53, 54], in string theory. In the braneworld picture, our reality is confined to a (spatially-three-dimensional) brane, inside a higher-dimensional “bulk” space. All standard model particles at low enough energy are trapped on the branes, while gravity, as a manifestation of space-time curvature, is free to propagate in all dimensions. The confinement is achieved by the dynamics of the corresponding theory. This paradigm removes the Planck-scaled condition imposed on the Kaluza-Klein picture. The size of the extra dimension(s) can be much larger than the Planckian size (and in some models it can even be infinite!) yet the ordinary particles are still trapped on the brane by a confinement mechanism. Now the question boils down to what kind of dynamical mechanism allows
such a trapping. In 1983, Rubakov and Shaposhnikov \[55\] (and similarly by Akama \[56\] and Visser \[57\]) proposed a simple field-theoretic model for the confinement mechanism. They argued that ordinary particles can be trapped inside a potential well, which is sufficiently narrow along the extra dimension(s) and flat along the brane. The properties of the well resembles domain walls. Indeed, Rubakov and Shaposhnikov showed that if the 5d bulk spacetime possesses a domain wall living in the fourth dimension, \textit{i.e.}, eq.\textnormal{(1.5)} in \(x^4\), then scalar and massless fermion fields can effectively be trapped\[29\]. This is enabled by the existence of a zero mode \[58\] in the spectrum of solutions. This zero mode is localized around the core of the wall while vanishing exponentially for large \(x^4\), mimicking our matter’s wavefunction. In this way, ordinary matter fields are localized only on the brane. The difference between this method and the Kaluza-Klein compactification scenario is that the latter treats all fields (including graviton) in the same manner with respect to compactification, while the braneworld picture singles out gravity (and perhaps some exotic fields, \textit{e.g.}, dilaton) as the only fields that are free to propagate in all dimensions. Since only graviton can dissipate into the extra dimensions, their sizes need not necessarily be small. Extra dimensions with size much larger than the Planck scale can still effectively be hidden from the 4d observer’s eye, while at the same time are testable.

More recent progress in the braneworld scenario indeed opens up the possibility that the extra dimensions can be much larger than the Planck scale (or even infinite) and can amount to explaining why gravity is so weak, \textit{i.e.},

\[29\]The localization of gauge fields is more non-trivial, as discussed in \[51\]. See references therein for some successful examples.
the hierarchy problem. See the discussions in [59][60] for large-compact extra dimensions, in [61][62] for non-compact extra dimensions, and in [63] for infinite extra dimensions.

1.6.3 Non-perturbative Instability in Kaluza-Klein Vacuum

In the Kaluza-Klein theory, the ground state is not $M_5$ (Minkowski space-time in 5 dimensions) but $M_4 \times S^1$. The original Lagrangian, eq. (1.50), is symmetric under the full 5-dimensional space-time symmetry, but the boundary condition imposed on the ansatz spontaneously breaks it. The low-energy physics lives on the $M_4 \times S^1$ ground state and the spectrum of particles are obtained by perturbation around this background.

One can ask whether this is a realistic ground state; i.e., whether $M_4 \times S^1$ is stable. In field theory we can infer the stability from energy consideration. A state is stable when its energy is minimum. In gravitational physics, however, this notion becomes more subtle. There is no unambiguous way to compare energies between two states. The concept of energy in general relativity depends on boundary conditions. Since $M_5$ and $M_4 \times S^1$ have different boundary conditions there is no meaningful comparison of energies between the two states.

Thus, to study the stability of Kaluza-Klein ground state it is essential to investigate the properties of the space-time itself. $M_4 \times S^1$ is classically (meta)-stable, i.e., linear-perturbation analysis shows no exponentially growing

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30 The meta-stability is due to the value of radion which remains unfixed by the compactification mechanism.
modes in the spectrum. How about quantum fluctuations? Since at present we have no complete understanding of quantum theory of gravity yet, we cannot address this question in full. However, we can still investigate it as far as semiclassical regime. We can ask whether this classically-stable vacuum is unstable against semiclassical decay [64]. Such an instability, if exists, would be of non-perturbative nature since it is mediated by quantum tunneling between two perturbatively-stable vacua.

1.6.3.1 Semiclassical Decay

Semiclassical decay (of false vacuum to true vacuum) has been studied extensively in the literature. One can refer to the classic papers by Coleman [65, 66, 67] as the pioneer in this topic (see also [68]). To illustrate it, let us consider a scalar field theory having potential with two minima: false and true vacua. The potential is depicted in Fig.(1.1). Let us denote the

![Figure 1.1](image)

Figure 1.1: The shape of a potential $V(\phi)$ with two slightly-non-degenerate minima, $\phi_{\pm}$. The higher one is false vacuum while the lower one is true vacuum.
false minima with \( \phi_+ \) and the lower one with \( \phi_- \). Classically, the state \( \phi_+ \) is stable. Under the small perturbations the field will only oscillate around the minimum. But quantum-mechanically it can decay to the lower minima \( \phi_- \) via barrier penetration (\textit{i.e.}, quantum tunneling). This false vacuum \( \phi_+ \) is said to be \textit{semi-classically} unstable. The tunneling occurs via bubble nucleation, much like bubble nucleation in boiled water. The difference is that the former is caused by quantum tunneling while the latter happens due to thermal fluctuations. Bubbles of true vacuum will nucleate in the ambient of false vacuum.

The tunneling (\textit{i.e.}, bubble nucleation) is mediated by an \textit{instanton}, \textit{i.e.}, a Euclidean\footnote{A space with metric signature \((+,+,+,-)\). This is achieved from Lorentzian signature by formal analytic continuation of the time coordinate, \( x^4 \equiv -ix^0 \).} solution of the corresponding field equation that renders the Action finite. Coleman called this solution \textit{"the bounce"}, \( \phi_b(x) \), satisfying

\[
\lim_{|x| \to \infty} \phi_b(x) \to \phi_+,
\]

where \(|x| \equiv \sqrt{x^2 + x_4^2}\), a \textit{"Euclidean distance"}. The name \textit{"bounce"} stems from the fact that the solution interpolates between the two vacua, much like a particle bouncing back and forth (with respect to the \textit{"Euclidean time"}) between two hills in an \textit{"inverted"}-potential description. If the energy difference between the two vacua, \( \epsilon \equiv V(\phi_+) - V(\phi_-) \), is small, we can employ thin-wall approximation in analyzing the bubble solutions. In this approximation, the bubble wall’s thickness (\( \rho \)) is much smaller compared to its radius (\( R \)), \( \rho \ll R \). The wall can be treated as a domain wall separating the false vacuum outside with the true vacuum inside.
The decay probability per unit time per unit volume is given by

\[ \Gamma/V = Ae^{-B/h}(1 + \mathcal{O}(h)), \tag{1.57} \]

where \( A \) is the over-all coefficient which depends on the quantum corrections and \( B \) is the finite (Euclidean) Action of the corresponding theory, \( B = S_E \).

The evolution of the bubble wall right after its materialization is given by the analytic continuation of the Euclidean solution back to the Lorentzian signature. The bubble wall grows and expands with velocity approaching the speed of light as it converts the false vacuum region into true vacuum. Physically the expansion is due to the prevailing competition of vacuum energy inside (which acts as a pressure) against the surface tension on the bubble (which tends to shrink it to minimize the tension).

\subsection*{1.6.3.2 Bubble of Nothing}

In the context of gravitational theories, (gravitational) instantons are the non-singular positive-definite metric solutions of the Euclidean (vacuum or coupled) Einsteins equations. In cosmology, gravitational instantons describe space-time transition via semi-classical mechanism (quantum tunneling).

Witten showed that such an instanton exists for the Kaluza-Klein vacuum; \( i.e., M_4 \times S^1 \) is semi-classically unstable. Moreover, he found that the true vacuum happens to be a “hole” in space-time, a bubble with no space-time inside; \( i.e., a \text{ bubble of nothing}. \)

The gravitational instanton that has the same asymptotic behavior as Kaluza-Klein vacuum can be obtained by analytic continuation of the black hole solution in 5d space-time, the Tangherlini solution. The Lorentzian
metric is\footnote{Here we follow the discussion of bubble of nothing in\cite{71}.}
\begin{equation}
    ds^2 = -\left(1 - \frac{\ell^2}{\rho^2}\right)dt^2 + \frac{d\rho^2}{1 - \frac{\ell^2}{\rho^2}} + \rho^2 \left(d\psi^2 + \sin^2\psi d\Omega^2\right).
\end{equation}

It is convenient to define a new radial coordinate,
\begin{equation}
    r^2 \equiv \rho^2 - \ell^2,
\end{equation}
such that the metric is re-expressed as
\begin{equation}
    ds^2 = -\frac{r^2}{r^2 + \ell^2} dt^2 + dr^2 + r^2 \left(d\psi^2 + \sin^2\psi d\Omega^2\right).
\end{equation}

Upon analytic continuations,
\begin{align*}
    t & \to i\ell y, \\
    \psi & \to it + \frac{\pi}{2},
\end{align*}
the bubble of nothing metric now reads\footnote{The bubble of nothing metric is expressed in a different gauge than in\cite{64}. See discussion in\cite{72}.}
\begin{equation}
    ds^2 = \frac{\ell^2}{1 + \frac{\ell^2}{\rho^2}} dy^2 + dr^2 + (r^2 + \ell^2) \left(-dt^2 + \cosh^2 t d\Omega^2\right).
\end{equation}

This metric is non-singular, provided the periodicity condition on $y$ is imposed, $0 \leq y \leq 2\pi$. This is a space into which the Kaluza-Klein vacuum decays. This space has the right asymptotic behavior, approaching Kaluza-Klein metric, $M_4 \times S^1$, for large $r$, where $M_4$ is written in a uniformly-accelerating gauge, the Rindler space.

At $r = 0$, the extra dimension degenerates to a point, and the space-time pinches off in a non-singular way. A hole is created in the manifold. A hole with non-singular boundary and nothing inside; no space-time. In the
Figure 1.2: A 4d conformal diagram for Bubble of Nothing. The interior of the hyperboloid does not exist. Space-time is only defined in the shaded region.

In the language of vacuum decay, we say that Kaluza-Klein vacuum is semi-classically unstable. Quantum fluctuations spontaneously form a bubble with no space-time inside. A bubble of nothing in the explicit sense. The induced-metric on the bubble wall is given by the $r$-slice of eq. (1.62), a 2 + 1 dimensional de Sitter space ($dS_3$) with size $\ell$, representing the growth of the bubble with time. Our 4d Kaluza-Klein world is unstable. Worse, this instability renders our world just disappears to nothing. Surely it poses a serious problem for any compactification scenario. Any realistic compactification theory must ensure that the probability of this decay is, at least, suppressed.
Chapter 2

Non-singular Braneworld

2.1 Cosmic p-Branes

With the progress of string theory, extra dimensions have become the paradigm in modern theoretical physics. String theory requires the existence of extra dimensions for its quantization consistency. Although its complete description as the theory of quantum gravity is yet to be established, in the low-energy limit string theory admits a rich variety of (non-perturbative) extended objects which, remarkably, can be identified as solitons \[73\, 74\]. These are solutions of higher-dimensional gravity coupled to \((d - p - 2)\)-form\(^1\) (and, in some cases, with dilaton). They bear the name: \(p\)-branes.

A class of extended black holes solutions, \(e.g.,\) black strings and black \(p\)-branes, were discovered and discussed in great length by Gibbons and Maeda \[75\] and by Horowitz and Strominger \[76\]. Two notable features of these solutions are that: (i) they lack \textit{boost}-symmetry, and (ii) they are unstable \[77\], except

\[^1\text{d is the total number of space-time dimensions, and p is the number of spatial dimensions living on the brane. This object is a generalization of Maxwell field strength tensor, } F_{\mu\nu},\text{ which is a 2-form} \]
when they are extremal \[78\]. Remarkably, the boost-symmetry is restored in
the extremal case. One may wonder whether these two properties are actually
related to one another: that the instability is caused by the failure of solutions
to be boost-symmetric.

Gregory \[79\] considered uncharged $p$-branes in $d$-number of space-time di-
mensions whose metrics respect the boost symmetry. She assumed the metric
be of the following form

$$ds^2 = B^2 \eta_{\mu \nu} dx^\mu dx^\nu - B^{-2n} dr^2 - C^2 d\Omega_{D-2}^2,$$

(2.1)

with $D = d - p$, and $i = 1, ..., p$ runs over the brane coordinates. Solving the
vacuum Einstein’s equations, she found a one-parameter family of solutions

$$B = \left( 1 - \left( \frac{r_0}{r} \right)^{D-3} \right)^{\frac{m}{p+n+1}},$$

$$C = r \left( 1 - \left( \frac{r_0}{r} \right)^{D-3} \right)^{\frac{1-m}{D-2}},$$

(2.2)

with

$$n = \frac{p + 1}{D - 3} + \frac{D - 4}{D - 3} \sqrt{\frac{(p + 1)(D + p - 2)}{D - 2}},$$

$$m = \frac{D - 4}{2(D - 3)} + \frac{D - 2}{2(D - 3)} \sqrt{\frac{(p + 1)(D - 2)}{D + p - 2}},$$

(2.3)

and $r_0$ an integration constant. These solutions can be regarded as the ex-
tremal case of the ones found in \[76\].

An unpleasant behavior of these solutions is that there is a naked singular-
ity at $r = r_0$. In the non-extremal case it is shown in \[76\] that the singularity
lies inside an event-horizon. The boost-symmetric condition forces the solu-
tions to be extremal, and as a result the singularity and the event-horizon

\(^2\)This is obtained by taking the appropriate limit of vanishing charge.
coincide. It seems that this naked singularity is the price to pay for being boost-symmetric.

### 2.2 Skyrme Branes

The naked singularity found in the Gregory’s solutions led her to conjecture that it can be avoided by putting matter fields at the origin. The core of the fields will smooth out the singularity at the origin. The question then is, what will be the appropriate choice of core model? The condition that space-time should asymptote to the vacuum solution, \( i.e., \) the metric should be asymptotically flat in the transverse directions, requires matter whose energy density falls-off rapidly outside the core. A natural candidate is a topological defect. Furthermore a condition that the brane be uncharged severely restricts the options. For the case of co-dimension-three brane a natural choice would be Skyrmions, \( i.e., \) the Skyrme branes.

Recently there have been many studies on topological defects with non-canonical kinetic terms \([80, 81, 82, 83, 84]\). Most of them discuss the differences introduced in the soliton solutions by the new kinetic term in the Lagrangian. Many of the original models already possess solitons in their spectrum of solutions, and so the additional term gives correction to the already-existing solitonic solutions. In contrast, the Skyrme model does not have any potential term, so the existence of the non-canonical kinetic term is essential, and not only a small correction to the model, for the stability of the solutions, as we have seen in the introductory chapter.

\[^{3}\text{t Hooft-Polyakov monopoles will not do the job since they leave an unbroken } U(1), \text{ making it charged.}\]
Einstein’s equations coupled to Skyrme field has been discussed extensively in the literature [85, 86, 87], either as black holes or regular solutions. Einstein-Skyrme black holes can provide a counter-example of the “No-hair Theorem” [88] by letting an uncharged black holes having scalar hair. The regular solutions can model compact stars (see, for example, [89]). Here in this dissertation we give a different interpretation of the solitonic solutions of Einstein-Skyrme equations in the context of braneworld cosmology.

2.2.1 The Einstein-Skyrme Model in 7D

The model is given by [90]

\[ S_{ES} = \int d^7X \sqrt{-g} \left( \frac{1}{2\kappa^2} R + \mathcal{L}_S \right), \]  

(2.4)

where \( R \) is the 7d Ricci scalar, \( \kappa^2 \equiv 1/M_{(7)}^5 \), with \( M_{(7)}^5 \) denoting the 7d Planck mass, and \( \mathcal{L}_S \) is the Skyrme Lagrangian density,

\[ \mathcal{L}_S = \frac{F_0^2}{4} Tr \left( L_A L^A \right) + \frac{1}{32e^2} Tr \left( [L_A, L_B][L^A, L^B] \right), \]  

(2.5)

with\(^4\) \( F_0 \) and \( e \) are two free parameters of the model with units of \([M]^{5/2}\) and \([M]^{-3/2}\), respectively. The left-chiral current \( L_A \) is defined

\[ L_A \equiv U^\dagger \partial_A U, \]  

(2.6)

with \( U \in SU(2) \).

We are interested in finding the smooth solution for a 3-brane that is spherically-symmetric along the transverse directions in the bulk. In this dissertation we will also restrict ourselves to the four-dimensional flat brane

\(^4\)\( F_0 \) here is the same as the pion decay constant \( F_\pi \) discussed in the introduction. In this braneworld model, however, the interpretation is different and we leave it as a free parameter.
solutions. Taking these constraints into account we can now write the most general metric which respects the corresponding symmetry, in the *isotropic gauge* as

\[ ds^2 = B^2(r)\eta_{\mu\nu}dx^\mu dx^\nu + C^2(r) \left( dr^2 + r^2d\Omega_2^2 \right). \]  
(2.7)

We impose the hedgehog ansatz for the chiral field; that is

\[ U(r) = \cos f(r) + i \left( \hat{\tau}^j \right) \sin f(r), \]  
(2.8)

with \( f(r) \) the profile function to be solved for. Within this ansatz, the Lagrangian density becomes

\[ \mathcal{L}_S = -\frac{F_0^2}{2} \left[ \frac{1}{C^2} \left( \frac{df}{dr} \right)^2 \left( 1 + \frac{2\sin^2 f}{e^2 F_0^2 C^2 r^2} \right) + \frac{\sin^2 f}{C^2 r^2} \left( 2 + \frac{\sin^2 f}{e^2 F_0^2 C^2 r^2} \right) \right]. \]  
(2.9)

It is convenient to rescale the radial coordinate \( eF_0 r \rightarrow x \) and define

\[ u \equiv \frac{1}{C^2} \left( 1 + \frac{2\sin^2 f}{C^2 x^2} \right), \]
\[ v \equiv \frac{\sin^2 f}{C^2 x^2} \left( 2 + \frac{\sin^2 f}{C^2 x^2} \right), \]  
(2.10)

to obtain

\[ \mathcal{L}_S = -\frac{e^2 F_0^4}{2} (uf'^2 + v), \]  
(2.11)

where “prime” denotes derivative with respect to \( x \).

Having simplified the Lagrangian, the Action for the Skyrme field becomes

\[ S_S = \int d^7 X \sqrt{-g} \mathcal{L}_S, \]
\[ = \frac{-2\pi F_0}{e} \int (uf'^2 + v) B^4 C^3 x^2 dxdy. \]  
(2.12)

---

\(^5\) We are unable to obtain numerical solutions in the Gregory’s gauge. However, later we will show that it is always possible to bring it into our isotropic gauge. Thus, the solutions are equivalent.

\(^6\) We use the mostly-positive metric signature, \( \eta_{\mu\nu} = diag(-1, 1, 1, 1) \).

\(^7\) We use notation \( y \) denote the (3 + 1) coordinates to avoid confusion with the *rescaled* radius \( x \).
It is now straightforward, by varying the Action with respect to $f(r)$, to obtain the equation of motion for the profile function,
\[
f'' = \frac{1}{2u} (uf'^2 + v_f) - \left[ B' + 3 \frac{C''}{C} + \frac{u'}{u} + \frac{2}{x} \right] f',
\] (2.13)
where
\[
u_f = \frac{\delta u(x)}{\delta f(x)},
\]
\[
u_f = \frac{\delta v(x)}{\delta f(x)}.
\] (2.14)

On the other hand, varying the Skyrme Action with respect to the metric tensor $g_{AB}$ yields
\[
T_{AB} = -\frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{AB}},
\] (2.15)
where $T_{AB}$ is the energy-momentum tensor for the chiral field given by
\[
T_{AB} = g_{AB} L - \frac{F_0^2}{2} Tr (L_A L^A) - \frac{1}{8e^2} g^{MN} Tr ([L_A, L_M][L_B, L_N]).
\] (2.16)

In our hedgehog ansatz and isotropic gauge, this becomes
\[
T_{\mu\nu} = -\frac{e^2 F_0^4}{2} (uf'^2 + v) \delta_{\mu\nu},
\]
\[
T_{xx} = \frac{e^2 F_0^4}{2} (uf'^2 - v),
\]
\[
T_{\theta\theta} = \frac{e^2 F_0^4}{2C^2} \left( -f'^2 + \frac{\sin^4 f}{C^2 x^4} \right),
\]
\[
T_{\varphi\varphi} = T_{\theta\theta}.
\] (2.17)

Using this form of energy-momentum tensor and our ansatz for the metric we obtain Einstein’s equations of the form
\[
G_{\mu\nu} = e^2 F_0^2 \delta_{\mu\nu} \left[ \frac{3B''}{BC^2} + \frac{2C''}{C^3} + \frac{3B'^2}{B^2 C^2} - \frac{C'^2}{C^4} + \frac{6B'}{BC^2 x} + \frac{4C'}{C^3 x} + \frac{3B'C''}{BC^3} \right]
\]
\[
= -\frac{\kappa^2 e^2 F_0^4}{2} (uf'^2 + v) \delta_{\mu\nu},
\]
41
\[ G_x = e^2 F_0^2 \left[ \frac{8B'}{BC^2x} + \frac{6B'}{B^2C^2} + \frac{2C'}{C^3x} + \frac{8B'C'}{BC^3} + \frac{C'^2}{C^4} \right] = \frac{\kappa^2 e^2 F_0^4}{2} \left( u f'^2 - v \right), \]

\[ G_\theta = e^2 F_0^2 \left[ \frac{4B''}{BC^2} + \frac{C''}{C^3} + \frac{6B'^2}{B^2C^2} - \frac{C'^2}{C^4} + \frac{4B'}{BC^2x} + \frac{C'}{C^3} \right] \]

\[ = \frac{\kappa^2 e^2 F_0^4}{2C^2} \left( -f'^2 + \frac{\sin^4 f}{C^2 x^4} \right), \]

\[ G_\varphi = G_\theta. \tag{2.18} \]

Equations (2.18) and (2.13) constitute the equations of motion for the 7d Einstein-Skyrme model consistent with the restrictions imposed by our ansatz.

### 2.2.2 Numerical Results

We would like to find solitonic solutions characterized by topological charge \( B \), defined in eq. (1.33). We will focus on \( B = 1 \). Using our ansatz, the invariant-charge can be written as

\[ B = \frac{\epsilon^{ijk}}{24\pi^2} \int Tr \left( L_i L_j L_k \right) C^3 r^2 dr d\Omega_2 \]

\[ = \frac{2}{\pi} \int \sin^2 f df \]

\[ = \frac{2}{\pi} \left[ \frac{f}{2} - \frac{\sin 2f}{4} \right]_{f(0)}^{f(\infty)}. \tag{2.19} \]

One can see that fixing the charge specifies the boundary conditions for \( f(r) \).

For \( B = 1 \) we need to impose the following boundary conditions

\[ f(0) = \pi, \quad f(\infty) \to 0. \tag{2.20} \]

Equations (2.18) and (2.13) appear to be hopeless to solve in closed forms. This is not surprising, because even in the flat space, eq. (1.41), no known analytic solution is found. We are then led to employ numerical methods to solve them. There are three Einstein’s equations and one chiral field equation.
for three unknown functions $B$, $C$, and $f$. This implies that not all Einstein’s equations are independent; one can be written as linear combinations of the other two. We choose to solve the $\mu\nu$- and $\theta\theta$-parts, along with the chiral field equations. The $xx$-part is used as a constraint.

We want to integrate our equations of motion starting from the core of the defect, so we still need to specify the conditions for the metric functions at the origin, $x = 0$. We expand the functions to be solved around the origin. Since we demand singularity-free, they should behave like power-series around the origin. The most general expressions are

\begin{align*}
B(x) & = B_0 + B_2 x^2 + \mathcal{O}(x^4), \\
C(x) & = C_0 + C_2 x^2 + \mathcal{O}(x^4), \\
f(x) & = \pi + f_1 x + f_3 x^3 + \mathcal{O}(x^4). \quad (2.21)
\end{align*}

Notice that there are no odd terms in the metric expansions while the chiral field’s expansion does not contain any even term. The odd terms in the metric are forbidden if we wish to make regular the invariant quantities: $R$, $R_{\mu\nu}R^{\mu\nu}$, and $R^{\alpha\beta\mu\nu}R_{\alpha\beta\mu\nu}$. The chiral profile function $f(x)$ does not have any even term since otherwise the field equation is not satisfied identically order by order. The coefficients in eq.\,(2.21) are found by plugging the expansion into the equations of motion. They are completely determined in terms of $f_1$, $B_0$, $C_0$, and $\hat{\kappa}^2 \equiv \kappa^2 F_0^2$. The lowest order are given by

\begin{align*}
B_2 & = \frac{B_0 f_1^4 \hat{\kappa}^2}{10 C_0^2}, \\
C_2 & = \frac{-f_1^2(5C_0^2 + 11 f_1^2)\hat{\kappa}^2}{40 C_0}, \\
f_3 & = \frac{-f_1^3(162 f_1^4 \hat{\kappa}^2 - 5 C_0^4(-16 + 3 \hat{\kappa}^2) + 5 C_0^2 f_1^2(8 + 9 \hat{\kappa}^2))}{600(C_0^4 + 2 C_0^4 f_1^2)}. \quad (2.22)
\end{align*}
The coefficients $f_1$, $B_0$, and $C_0$ are left undetermined. We solve the equations numerically using shooting method, and those parameters become the *shooting parameters*; i.e., that is, we adjust the values of $f_1$, $B_0$, and $C_0$ such that the asymptotic solutions satisfy

$$f(\infty) \to 0,$$

$$B(\infty) = C(\infty) = 1.$$  \hspace{1cm} (2.23)

We show in Figs.(2.1)-(2.3) some results found by this method. Since the equations contain terms inversely proportional to $x^n$ it is immediately clear that we cannot integrate them numerically from $x = 0$ as those terms will blow up. Therefore we pick a small enough number $0 < x_0 \ll 1$ to start the integration. The smaller $x_0$ is the more accurate the result is. In our calculation we choose $x_0 = 10^{-2}$. Also, in practice it is impossible to integrate all the way to infinity. There should be a large enough value of the integration range, $x = x_{\text{max}}$, up to where we integrate our equations whose results we can trust should it be extrapolated to infinity. Here we set $x_{\text{max}} = 10^4$. What we show on the metric figures are the regular behavior at the origin. The metric coefficients are non-singular. At large $x$ (which is still much less than our $x_{\text{max}}$) our solutions rapidly go to the asymptotically-flat case. We see no difficulty in going to smaller $x_0$ and greater $x_{\text{max}}$. This results in more accurate solutions, at the price of higher numerical fine-tuning.

We integrate the equations for different values of $\hat{\kappa}$ (or, equivalently, $\kappa$). For $\kappa = 0$ the system decouples; gravity is dictated by the vacuum Einstein’s equations, while the Skyrme model is reduced to that in flat space. For $\hat{\kappa} \neq 0$ we found, as in the case of the 4d Einstein-Skyrme system [85, 86], that there
exists a critical value of $\hat{\kappa}$, $\hat{\kappa}_{\text{crit}}$, beyond which no non-singular solutions are possible. Our numerical investigation reveals that this critical value is around the order of $\hat{\kappa}_{\text{crit}}^2 \sim 0.05$. It is not yet clear what type of object one should have that enables this larger values of coupling constant. On simple heuristic argument \[86\] one can show that, since the mass of branes is of order $M \sim F_0/e$, its corresponding Schwarzschild radius is around $R_{\text{Schw}} = 2GM \sim \kappa/e$. Thus, for fixed value of $F_0$ and $e$, higher $\hat{\kappa}$ makes $R_{\text{Schw}}$ comparable to the defects core, $R \sim 1/eF_0$. Therefore, on gravitational grounds, a configuration with $\hat{\kappa}$ higher than some critical value will collapse forming a black brane; a *Skyrme black brane*. Another possibility is by relaxing the staticity of the metric, *i.e.*, allowing it to be time-dependent. This has a natural interpretation of an *inflating brane*. This is certainly plausible since it is shown elsewhere that an inflating brane can regularize its singularity \[91\,92\,93\]. However we should point out that the scenario of *topological inflation*\[94\,95\] does not

---

\[8\] The idea that inflation can occur in the core of topological defects and that the topo-
seem to apply here since Skyrme model does not have any potential energy.

Figure 2.2: Typical form of the warp factor $B(r)$ (for $\kappa^2 = 1/25$).

Figure 2.3: Typical form of the function $C(r)$ (for $\kappa^2 = 1/25$).

Another interesting result is that, for each value of $\hat{\kappa} < \hat{\kappa}_{\text{crit}}$, we have two

logical stability forces the field to stay on the potential hill, thus making it eternal even at classical level.
branches of solutions, *the lower* and *upper branches* (see Figs. 2.3). They correspond to different values of shooting parameters for any given \( \hat{\kappa} \). This non-uniqueness solution is not found in the case of Skyrmions in flat space; each collection of shooting parameters uniquely determines the solution. This suggests that the gravity can split Skyrmion solutions. One of the branch is unstable and this corresponds to the extreme limit of Skyrme model. This is also another feature that is shared by the 4d system [86]. In Fig. 2.4 we show an example of two branches of solution of the profile function \( f(r) \) for \( \hat{\kappa}^2 = 1/100 \). Note that the term “upper” refers to the higher value of the shooting parameter \(-f_1\). So what we refer as the *upper branch* is actually the lower solution\(^9\). The numerical solutions we found asymptotically approach

![Figure 2.4: A branch of solutions of the profile function \( f(r) \) for \( \hat{\kappa}^2 = 1/100 \). Be advised that the lower solution (the red plot) is what we call the *upper branch* in this dissertation.](image)

\(^9\)In the color version, the red plot. We apologize for the confusion of terminology.
flat space, and it is therefore possible to identify the form of the energy-momentum tensor sourcing this metric, in a similar way to what one does in 4d case [96]. Following [97, 98] we obtain

\[ T_{\mu\nu}^{ADM} = \lim_{\hat{r} \to \infty} \frac{1}{2\hat{\kappa}^2} \oint \hat{r}^i \left[ \eta_{\mu\nu} \left( \partial_i h^\sigma_{\sigma} + \partial_i h^j_j - \partial_j h^i_i \right) - \partial_i h_{\mu\nu} \right] r^2 d\Omega_2, \]  

(2.24)

with \( h_{AB} \equiv g_{AB} - \eta_{AB} \) the perturbation of our metric around the flat space, \( \hat{r}^i \) the radial unit vector in the transverse three-dimensional space, \( \mu, \nu, \sigma = 0, ..., 3, \) and \( i, j = 4, 5, 6. \) Using our expression of our ansatz we obtain

\[ T_{\mu\nu}^{ADM} = \frac{4\pi}{\hat{\kappa}^2} \eta_{\mu\nu} \lim_{\hat{r} \to \infty} \left[ r^2 (3BB' + 2CC') \right], \]

\[ \equiv -T_{ADM} \eta_{\mu\nu}. \]  

(2.25)

It is clear that we can read off the value of the 3-brane source tension from the behavior of the metric. In Table 2.1 we show our numerical results for the tension \( T^{ADM} \) and the metric parameter \( r_0. \) It is shown from the table that the upper branch has higher tension, thus signaling the instability of the solutions.

2.2.3 Cosmic 3-Branes in the Isotropic Gauge

In the language of ref. [99], our solutions can be classified as topologically non-trivial \textit{thick branes}. Our branes are the smooth version of the thin wall limit approximation, and the inclusion of a defect core introduces thickness to the brane, (roughly) characterized by the defect width.

We shall now show that our solutions do correspond to the smooth version of the Gregory’s thin wall vacuum solutions [79]. The metric solutions of the
Figure 2.5: Two fundamental branches of solitonic solutions. The shooting parameter $f_1$ (vertical) is plotted as a function of $\hat{\kappa}^2$ (horizontal). We can see how the branches merge at $\hat{\kappa} = \hat{\kappa}_{\text{crit}}$. On the other hand for smaller $\kappa$ their splitting distance increases, and in the limit of $\hat{\kappa} \to 0$ the upper branch goes to infinity while the lower branch recovers the flat space case, suggesting that the former is unstable.
Table 2.1: We show the values of the brane tension $T^{ADM}$ (in units of $F_0/e$) and the parameter $r_0$ (in units of $eF_0$).

<table>
<thead>
<tr>
<th>$\tilde{\kappa}^2$</th>
<th>lower branch</th>
<th>upper branch</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$r_0$</td>
<td>$T^{ADM}$</td>
</tr>
<tr>
<td>0.01</td>
<td>0.071614</td>
<td>71.1456</td>
</tr>
<tr>
<td>0.02</td>
<td>0.139140</td>
<td>69.1149</td>
</tr>
<tr>
<td>0.03</td>
<td>0.202338</td>
<td>67.0048</td>
</tr>
<tr>
<td>0.04</td>
<td>0.260364</td>
<td>64.6652</td>
</tr>
<tr>
<td>0.048</td>
<td>0.303528</td>
<td>62.8214</td>
</tr>
<tr>
<td>0.0485</td>
<td>0.306311</td>
<td>62.7438</td>
</tr>
<tr>
<td>0.0487</td>
<td>0.307273</td>
<td>62.6824</td>
</tr>
<tr>
<td>0.0488</td>
<td>0.308737</td>
<td>62.8519</td>
</tr>
<tr>
<td>0.0489</td>
<td>0.308155</td>
<td>62.6052</td>
</tr>
</tbody>
</table>
vacuum Einstein's equations she found are in the following form

\[
\begin{align*}
\frac{ds^2}{\sqrt{10}} &= \eta_{\mu\nu}dx^\mu dx^\nu + \left(1 - \frac{r_0}{r}\right)^{-\frac{4}{\sqrt{10}}}dr^2 + \left(1 - \frac{r_0}{r}\right)^{1-\frac{4}{\sqrt{10}}}r^2d\Omega^2_2,
\end{align*}
\]

(2.26)

while our numerical solutions are found within the *isotropic* gauge, eq.(2.7). However, it is always possible to transform Gregory’s gauge into the *isotropic* form. In this gauge, the 3-brane vacuum solutions become

\[
\begin{align*}
\frac{ds^2}{\sqrt{10}} &= \left(\frac{4\tilde{r} - r_0}{4\tilde{r} + r_0}\right)^{\frac{2}{\sqrt{10}}}\eta_{\mu\nu}dx^\mu dx^\nu + \left(\frac{4\tilde{r} - r_0}{4\tilde{r} + r_0}\right)^4 \left(\frac{4\tilde{r} + r_0}{4\tilde{r} - r_0}\right)^{2+\frac{8}{\sqrt{10}}} (dr^2 + \tilde{r}^2d\Omega^2_2).
\end{align*}
\]

(2.27)

We can use the asymptotic form of this gauge to calculate the ADM energy-momentum tensor, eq.(2.25),

\[
T_{\mu\nu}^{ADM} = -\frac{\sqrt{10\pi}}{\tilde{r}^2}r_0\eta_{\mu\nu},
\]

(2.28)

which depends on the metric parameter $r_0$ that completely characterizes the 3-brane metric solution. Note that the Gregory’s vacuum solutions (2.26) are plagued with naked singularities located at $\tilde{r} = r_0$ and at $\tilde{r} = 0$, while our solutions are smooth everywhere, free from any singularity. We will show that the analytic solutions are in fact a good approximation to our numerical results in the asymptotic region where $\tilde{r} \gg r_0$, but start deviating from them as $\tilde{r} \sim r_0$.

To do this, we first identify the value of $r_0$ by looking at the asymptotic form of the ADM energy-momentum tensor using eqs.(2.25) and (2.28). Once it is fixed, it singles out a particular member within the family of 1-parameter solutions, eq.(2.27). Using this parameter we plot the functions $B(r)$ and $C(r)$ (Figs.(2.6)-(2.7)) and compare them with the same functions found numeri-
cally. As we see the asymptotic form of the two functions, the numerical and the analytic vacuum solutions match perfectly.

Figure 2.6: Comparison between the numerical and analytic vacuum solutions of $B(r)$ (for $\kappa^2 = 1/25$).

Figure 2.7: Comparison between the numerical and analytic vacuum solutions of $C(r)$ (for $\kappa^2 = 1/25$).
Therefore we can conclude that we do prove Gregory’s conjecture by showing that the naked singularity in the cosmic 3-brane can be smoothed out by having an uncharged defect core, a Skyrme brane.

### 2.2.4 Pure Skyrme Limit

The existence of two branches of solutions can be analyzed by studying their behavior at \( \hat{\kappa} \to 0 \). From Fig.\( (2.4) \) we can see that in that limit the upper branch \( f_{1u} \) goes to infinity. Recall that \( \hat{\kappa} \) is defined as \( \hat{\kappa} = \kappa F_0 \). Thus, the limit \( \hat{\kappa} = 0 \) corresponds to, either: (i) \( \kappa = 0 \), or (ii) \( F_0 = 0 \). As we already discussed above, the case \( \kappa = 0 \) corresponds to the decoupling of Einstein-Skyrme system. The Skyrmion lives in the ordinary 4d flat space, and its profile function \( f(r) \) is described by the lower branch solution. The more interesting case is when \( F_0 = 0 \). This is the limit of vanishing the first term in the Skyrme Lagrangian (eq.(1.36)). In this limit, the system is still gravitationally coupled but now the Skyrme sector is dominated by the second term; the Skyrme term. This limit is called the pure Skyrme limit. It is the limit discussed in ref. \[100\].

In ref. \[86\] it was shown that in the case for vanishing \( F_0 \) the coupled equations are identical to the static-spherically-symmetric magnetic Einstein-Yang-Mills equations. The Einstein-Yang-Mills equation is known to have static globally-regular (i.e., non-singular, asymptotically-flat) lump-like solutions, dubbed \textit{Bartnik-McKinnon} solitons \[101\]; a surprising result since neither the vacuum Einstein nor the pure Yang-Mills in (3 + 1)-dimensions can support such solutions\[10\]. Stability analysis shows that \textit{Bartnik-McKinnon}...
solitons are unstable.

It was shown by Wospakrik \cite{104} that the pure Skyrme limit reduces to the pure Yang-Mills system in some particular choice of gauge. Thus, it is natural to conjecture that the upper branch of the Einstein-Skyrme model is also unstable.\footnote{In 4d case this is verified by the discovery of one negative mode in the perturbative analysis of the upper branch solutions \cite{86}.}

\section{2.3 Non-singular Arbitrary co-dimensional Branes}

Gregory’s metric solutions, eq.\eqref{2.2}, is valid for any arbitrary dimensions. However, the defect core needed to smooth it out, \textit{i.e.}, the Skyrmion, is only stable in three spatial dimensions. Derrick’s theorem forbids static domain walls, strings, and magnetic monopoles to live in spatial dimensions higher than three. In order to regularize the naked singularities in higher dimensional thin-wall solutions, we need to find a stable uncharged higher-dimensional defect. In particular, we want to find the generalization of Skyrme model in more than (3 + 1) dimensions, \textit{i.e.}, an \textit{extended} Skyrmion.

From Derrick’s theorem we can learn that one way of obtaining a stable defect is by having higher order derivative terms. The Skyrme term is introduced for this reason. The same chain of reasoning suggests to us that in order to proceed to higher dimensions we need derivative terms higher than the order of Skyrme term. Skyrmions in (3 + 1) dimensions are stabilized by the Skyrme term, which is 4\textit{th} order in derivatives. We need (at least) an additional term system. See also \cite{102,103}. The existence of static solutions in the coupled Einstein-Yang-Mills system is possible because the attractive force of gravity is compensated by the repulsive Yang-Mills force.
which is sixth-order\textsuperscript{12} in derivatives to obtain extended Skyrmions in\textsuperscript{13} 5d.

The inclusion of sixth-order term in the Skyrme Lagrangian is discussed in refs. \cite{105,106}, where they introduce a term

\[ \mathcal{L} = c_6 \text{Tr} \left( [L_\mu, L_\nu][L_\nu, L_\lambda][L_\lambda, L_\mu] \right), \tag{2.29} \]

in the Lagrangian, with \( c_6 \) is a coefficient parametrizing the sixth-order term. It is built by contracting the topological charge \textsuperscript{14}. This particular form is chosen because it is the only sixth-order term which is Lorentz-invariant and produces equations of motion involving time-derivatives not higher than second order \cite{105}.

While adding the term \textsuperscript{(2.29)} in the Lagrangian enables one to obtain a static defects in (up to) \((d = 5 + 1)\) dimensions, obviously it is not sufficient to prevent it against collapse in \(d > 5\). Following the logic, we are led to introduce higher order terms every time we go one dimensional higher. This clearly is not an aesthetic approach, since: (i) we have to introduce the terms by hand, and (ii) we will have more and more free parameters characterizing each term which is (most likely to be) independent of one another. It would be nice if we can generate any arbitrary higher order terms in a natural way without introducing more free parameters. Furthermore, if we recall the existence of Skyrme term is also ad hoc. There is no underlying fundamental principle dictating such a term should exist, other than to have a stable solution. This leads people to find a generalization of the Skyrme term that can generate any higher order possible. Fujii et al \textsuperscript{107,108} proposed such a generalization that

\textsuperscript{12}The odd-power terms will not preserve Lorentz invariance. With this sixth-order term, we can have stable static solutions with or without the Skyrme term.

\textsuperscript{13}The sixth-order term can support stable extended Skyrmions up to \(d = 5 + 1\).

\textsuperscript{14}The topological charge \( B^\mu \sim \epsilon^{\mu
u\alpha\beta} L_\nu L_\alpha L_\beta \) is a natural higher-order generalization of the form \([L_\mu, L_\nu]\) in the Skyrme term.
works in any higher even dimensions. Another similar proposal generalizing the term (2.29) can also be found in ref. [109].

One attractive scenario to generate higher order terms is by considering the Born-Infeld-type\textsuperscript{15} Lagrangian. Dion et al\textsuperscript{[112]} consider the Born-Infeld-type of the Skyrme term

\begin{equation}
\mathcal{L} = -\frac{F_2^2}{4} Tr (L_\mu L^\mu) + \beta^2 \left( \sqrt{1 + \frac{1}{16 \beta^2 e^2} Tr [L_\mu, L_\nu]^2} - 1 \right),
\end{equation}

(2.30)

where \(\beta\) is the Born-Infeld parameter with dimension \([\ell]^{-4}\), which in large limit (i.e., corresponding to the low-energy limit), \(\beta^2 \to \infty\), reduces the Lagrangian into the ordinary Skyrme model, eq.(1.36). The DBI-type Lagrangian is convenient because, upon Taylor expansion on \(\beta\), we can generate arbitrary higher-order terms. This fits with our need to regularize the general Gregory’s cosmic p-branes.

However, in this work we do not employ the model (2.30). Instead, we are considering a simpler toy model, the so-called chiral Born-Infeld theory\textsuperscript{[113]} where, instead of the Skyrme term, we put the non-linear sigma model term under the DBI-form. The original chiral DBI theory has the following Lagrangian\textsuperscript{[113]}

\begin{equation}
\mathcal{L} = -F_2^2 Tr \beta^2 \left( 1 - \sqrt{1 - \frac{1}{2 \beta^2} L_\mu L^\mu} \right);
\end{equation}

(2.31)

the existence of Skyrme term is avoided altogether. At \(\beta \to \infty\) the Lagrangian is reduced to

\begin{equation}
\mathcal{L} \sim -\frac{F_2^2}{4} Tr L_\mu L^\mu,
\end{equation}

(2.32)

the ordinary non-linear sigma model. The advantage of this toy model is that

\textsuperscript{15}Also known as Dirac-Born-Infeld (DBI). It was proposed in 1934 as a non-linear theory of ELectrodynamics by Born and Infeld\textsuperscript{[110]} and later in 1962 Dirac\textsuperscript{[111]} showed that such a theory can be quantized.
it does not even require the Skyrme term to stabilize the defect. The DBI-form is sufficient to render the texture defects, eq. (1.30), stable. The Skyrme term is ad hoc, while the higher-order derivative terms in Lagrangian (2.31) can be arbitrarily generated through Taylor expansion. In this sense, it is more natural to generalize higher-dimensional stable texture defects within the chiral Born-Infeld model.

### 2.3.1 Extended Chiral-DBI Solitons in Flat Space-time

We wish to obtain extended solitons from the Lagrangian (2.31) in \( N \)-dimensions. For our purposes, it is more convenient to re-write the ch-DBI Lagrangian in terms of non-linear sigma model\(^{17}\) (eq. (1.30)),

\[
\mathcal{L} = \beta_N^2 \left[ \sqrt{1 + \frac{1}{\beta_N^2} \partial_\mu \Phi^i \partial^\mu \Phi^i} - 1 \right],
\]

where \( i = 1, 2, ..., d \). The non-linear constraint \( \Phi^i \Phi^i = 1 \) spontaneously breaks the symmetry of the theory, \( SO(N+1) \to SO(N) \), where \( N \equiv d - 1 \). From homotopy theory we know that it defines the vacuum manifold \( \mathcal{M} = SO(N+1)/SO(N) \cong S^N \), whose \( n \)-th homotopy group is non-trivial, \( \pi_N(S^N) = \mathbb{Z} \).

This topological invariant number manifests itself in the topological current

\[
\sqrt{-g} j^\mu = \frac{1}{12\pi^2} \epsilon_{\alpha_1 \alpha_2 \cdots \alpha_{N-1} \alpha a_1 a_2 \cdots a_N} \Phi^{a_1} \partial_{\alpha_1} \Phi^{a_2} \partial_{\alpha_2} \phi_{a_3} \cdots \partial_{\alpha_{N-1}} \Phi^{a_N}. \quad (2.34)
\]

Having examined the non-trivial topology of the vacuum manifold, we should ask ourselves whether there exists a static defect in its spectrum. Let

\(^{16}\)This section is based on our unpublished work, which is going to be published soon after the dissertation is completed.

\(^{17}\)Notice that we express the coupling constant as \( \beta_N \), since it is dimensionful with dimension \( \beta_N \sim [\ell]^{-(N+1)} \).
us consider the static energy

\[ E = \left( 1 - \sqrt{1 - E_2} \right), \quad (2.35) \]

where, for simplicity, we set\[^{18}\] \( \beta_N = 1 \), \( E_2 \equiv \partial_i \phi_a \partial^i \phi_a \), and it is to be understood that the right hand side is under the integral sign. Under the scale transformations, the energy rescales as

\[ E_\lambda = \lambda^{-N} \left( 1 - \sqrt{1 - \lambda^2 E_2} \right), \quad (2.36) \]

Conditions

\[
\begin{align*}
\frac{dE}{d\mu} \bigg|_{\mu=1} &= 0, \\
\frac{d^2E}{d\mu^2} \bigg|_{\mu=1} &> 0, \quad (2.37)
\end{align*}
\]

from Derrick’s theorem (see Appendix B) implies

\[
\begin{align*}
\frac{E_2}{\sqrt{1 - E_2}} - N(1 - \sqrt{1 - E_2}) &= 0, \\
\frac{E_2}{(1 - E_2)^{3/2}} + \frac{(1 - 2N)E_2}{\sqrt{1 - E_2}} + N(N + 1)(1 - \sqrt{1 - E_2}) &\geq 0, \quad (2.38)
\end{align*}
\]

respectively. It is immediately clear that there is no stable defect for\[^{19}\] \( N < 3 \).

On the other hand, one can prove that for \( N \geq 3 \) there always exists a non-zero positive value of \( E_2 \) such that the defects obtain a natural scale; \( i.e., \) the possibility of having static and stable defects are not ruled out in any higher dimensions.

Armed with this possibility we consider a hedgehog ansatz\[^{20}\], \( i.e., \)

\[
\Phi_i = (\cos \tilde{\theta}_1, \sin \tilde{\theta}_1 \cos \tilde{\theta}_2, \sin \tilde{\theta}_1 \sin \tilde{\theta}_2 \cos \tilde{\theta}_3, \ldots), \quad (2.39)
\]

\[^{18}\]0 < \( E_2 < 1 \) to give a finite and real energy.
\[^{19}\]The \( N = 2 \) case remains conformally-invariant, which we do not investigate here.
\[^{20}\]In this dissertation we only consider \( B = 1 \) topological charge.
with

\[ \tilde{\theta}_1 = \alpha(r), \]
\[ \tilde{\theta}_j = \theta_j, \quad j = 2, ..., N - 1, \quad (2.40) \]

where \( \alpha(r) \) is the chiral angle and \( \theta_j \) are the angular coordinates of an \( N \)-sphere. The Lagrangian becomes

\[ \mathcal{L} = \beta^2_N \left[ \sqrt{1 - K} - 1 \right], \quad (2.41) \]

where

\[ K \equiv \frac{1}{\beta^2_N} \left( \alpha^2 + \frac{(N - 1) \sin^2 \alpha}{r^2} \right), \quad (2.42) \]

where “primes” denotes derivative with respect to \( r \). This gives the \((N + 1)\)-dimensional Action

\[ S = \int d^{N+1} X \, \mathcal{L}, \]
\[ = \Omega_{N-1} \int dt \, dr^{N-1} \, \mathcal{L}, \quad (2.43) \]

where \( \Omega_{N-1} \) is the hypersurface area of an \((N - 1)\)-sphere. The Least Action Principle yields the equation for the chiral angle \( \alpha(r) \),

\[ \left[ \frac{\alpha' r^{N-1}}{\sqrt{1 - K}} \right]' - \frac{(N - 1) \sin 2\alpha}{2r^{3-N} \sqrt{1 - K}} = 0. \quad (2.44) \]

For \( d = 3 + 1 \), the solutions\(^{21}\) are given in [113].

We solve the eq. (2.44) numerically and found solutions satisfying the boundary conditions (2.23), up to \( d = 8 + 1 \) (Fig. (2.9)). They are the extended solitons in Chiral DBI theory.

For each number of spatial dimensions, the solution depends on one free parameter, the coupling constant \( \beta_N \). It characterizes the size of the defects.

\(^{21}\)In Fig. (2.8) we reproduce the ordinary \( d = 3 + 1 \)-solutions.
Figure 2.8: Chiral DBI soliton (with $\beta_3 = 1$) in flat $d = 3 + 1$. Here we reproduce the result of [113].

In Fig.(2.10) we show solutions for various $\beta_3$ (in $d = 3 + 1$). For $\beta_3 > 1$ the theory is weakly-coupled, and upon Taylor expansion the Lagrangian (2.33) reduces to the ordinary non-linear sigma model. On the other hand, for $0 < \beta_3 \leq 1$ we cannot neglect the higher-order expansion terms. The theory becomes strongly-coupled. One can see from Fig.(2.10) that as $\beta_3$ grows, the defects become thinner. We can understand it from the fact that for large $\beta_3$ it effectively reduces to an ordinary texture defect, and texture does not possess a natural scale; it is unstable against collapse to trivial vacuum configuration. As $\beta_N \to \infty$ the defect is infinitely thin (vacuum).

As $\beta_3$ becomes smaller the defects become thicker. It is easy to understand from the fact that the defect width is inversely proportional to $\beta_3^2$,

$$r_{\text{chiral}} \sim \beta_N^{-2}.$$  \hspace{1cm} (2.45)
Figure 2.9: One typical extended chiral DBI soliton (with $\beta_8 = 1/2$) in flat $d = 8 + 1$.

Figure 2.10: Chiral DBI solitons for various values of $\beta_3$ in flat $d = 3 + 1$. 
The size of (extended) defects for a fixed value of $\beta_N$ also varies as one goes to higher-dimensions. Fig. (2.11) shows extended defects for in several higher-dimensions. One can see that as the dimensions $d$ increases the defects become thicker while at the same time falls off faster asymptotically.\footnote{The defects are more localized.} The thickness behavior can be inferred from the fact that $\beta_N$ is a dimensionful parameter. Eq.(2.45) implies that the defect width increases as one goes to higher-dimensions.

The asymptotic fall-off behavior can easily be understood by studying the asymptotic expansion of eq.(2.44). At large radius, the chiral angle $\alpha(r)$ goes like

$$\alpha(r) \to \frac{b}{r^N - 1}, \quad (2.46)$$

with $b$ some numerical constant. As the dimensions increase the chiral angle falls off faster.

### 2.3.2 Chiral DBI Branes

Having established the solitonic solutions in higher-dimensional flat case, we would like to know whether they can be good model for non-singular $p$-brane cores. Consider the generalized isotropic gauge ansatz, eq.(2.7), in $d$-dimensions\footnote{$d=N+p+1.$}.

$$ds^2 = B(r)^2 \eta_{\mu\nu} dx^\mu dx^\nu - H(r)^2 (dr^2 + r^2 d\Omega_{N-1}^2), \quad (2.47)$$

where

$$d\Omega_{N-1}^2 \equiv \sum_{i=1}^{N-1} \Upsilon_i(\theta_{j<i}) d\theta_i^2, \quad (2.48)$$
Figure 2.11: Extended chiral DBI solitons (with $\beta_N = 1/2$) in various dimensions ($d = (3 + 1), (4 + 1), (5 + 1), \text{ and } (6 + 1)$).
with $\Upsilon_1 \equiv 1 \quad \text{and} \quad \Upsilon_{i>1} \equiv \sin^2 \theta_1 \prod_{j=2}^{i-1} \sin^2 \theta_j$, \hfill (2.49)

and $N$ is the number of the codimensions of the brane, on which the chiral DBI only depend.

Let us derive the Einstein’s equations for this isotropic gauge. The non-zero components of Christoffel Symbols are

\[
\Gamma_{\mu\nu}^r = \frac{BB'}{H^2} \eta_{\mu\nu}, \\
\Gamma_{r\nu}^\mu = \frac{B'}{B} \delta^\mu_\nu, \\
\Gamma_{rr}^r = H', \\
\Gamma_{\theta_i\theta_i}^r = -\Upsilon_i(\theta_{j<i}) \left[ \frac{H'^2}{H} + r \right], \\
\Gamma_{r\theta_i}^{\theta_i} = \frac{H'}{H} + \frac{1}{r}, \\
\Gamma_{\theta_k\theta_k|k>i}^{\theta_i} = -\frac{1}{2} \sin 2\theta_i \prod_{j=i+1}^{k-1} \sin^2 \theta_j, \\
\Gamma_{\theta_k\theta_k|k>i}^{\theta_k} = \frac{1}{\tan \theta_i}, \quad \text{(2.50)}
\]

from which the Ricci tensor components read

\[
R^0_0 = \left[ \frac{B''}{B} + p \left( \frac{B'}{B} \right)^2 + (N - 2) \frac{B'H'}{BH} + (N - 1) \frac{B'}{Br} \right], \\
R^r_r = \left[ (p + 1) \frac{B''}{B} + (N - 1) \frac{H''}{H} + (N - 1) \frac{H'}{Br} - (p + 1) \frac{B'H'}{BH} - (N - 1) \left( \frac{H'}{H} \right)^2 \right], \\
R^\theta_\theta = \left[ \frac{H''}{H} + (N - 3) \left( \frac{H'}{H} \right)^2 + (2N - 3) \frac{H'}{Br} + (p + 1) \frac{B'H'}{BH} + (p + 1) \frac{B'}{Br} \right]. \quad \text{(2.51)}
\]

The vacuum Einstein equation for zeroth component, $R^0_0 = 0$, can be inte-
grated to give
\[
[(B^{p+1})'H^{N-2}r^{D-1}]' = 0. \quad (2.52)
\]

By appropriately identifying the constant of integration of equation above, we could try an ansatz\(^{24}\) for \(B(r)\), generalizing our isotropic solution for \(p = N = 3\) \([90]\),:
\[
B(r)^{p+1} = \left(\frac{1 - (\frac{r_o}{r})^{N-2}}{1 + (\frac{r_o}{r})^{N-2}}\right)^m, \quad (2.53)
\]
with \(r_o\) integration constant. This yields
\[
H(r) = \left(\frac{1 + (\frac{r_o}{r})^{N-2}}{1 - (\frac{r_o}{r})^{N-2}}\right)^{\frac{m}{N-2}} \left[1 - \left(\frac{r_o}{r}\right)^{2(N-2)}\right]^{\frac{1}{N-2}}, \quad (2.54)
\]
which immediately solves \(R^6_\theta = 0\). The only left parameter to determine, \(m\), can be expressed in terms of variables \(p\) and \(N\) by plugging (2.53) and (2.54) into the remaining equation, \(R^r_r = 0\), which then, after some algebra, gives
\[
m = \sqrt{\frac{(p + 1)(N - 1)}{(N + p - 1)}}. \quad (2.55)
\]

Eqs (2.53) and (2.54), supplemented with (2.55), then completely solve the vacuum Einstein equations in isotropic form (see also \[114\]).

The matter-sector Lagrangian still takes the form of (2.41), with \(K\) now redefined as
\[
K \equiv \frac{1}{\beta_N^2} \left(\frac{\alpha'^2}{H^2} + \frac{(N - 1) \sin^2 \alpha}{H^2 r^2}\right). \quad (2.56)
\]
The equation for the chiral angle now becomes
\[
\left[\frac{A^{p+1}H^{N-2}r^{N-1}\alpha'}{\sqrt{1 - K}}\right]' - \frac{(N - 1)A^{p+1}H^{N-2}r^{N-3} \sin 2\alpha}{2\sqrt{1 - K}} = 0. \quad (2.57)
\]

Armed with the energy-momentum tensor
\[
T_{AB} = \frac{2}{\sqrt{g}} \frac{\delta S}{\delta g^{AB}}
\]
\(\text{---}\)\(^{24}\)Inspired by the derivation in [79]

65
\[
\frac{2}{\sqrt{g}} \left( -\frac{1}{2} g_{AB} \mathcal{L} + \sqrt{-g} \frac{\delta \mathcal{L}}{\delta g^{AB}} \right) \\
= -g_{AB} \mathcal{L} + \left( \frac{\partial_A \phi_i \partial_B \phi_i}{1 + \frac{1}{\beta_N} \partial_C \phi_i \partial^C \phi_i} \right)^{1/2}, \tag{2.58}
\]

which yields

\[
T_0^0 = -\mathcal{L}, \\
T_r^r = -\mathcal{L} - \frac{\alpha^2}{H^2 \sqrt{1 - K}}, \\
T_\theta^\theta = -\mathcal{L} - \frac{\sin^2 \alpha}{H^2 r^2 \sqrt{1 - K}}, \tag{2.59}
\]

we are ready to solve the coupled Einstein’s equations along with (2.57).

Figure 2.12: Typical form of the profile function $\alpha(r)$ (with $\beta_4 = 1/2$) in $d = 4 + 1$.

We solve the equations numerically, and show some of the defects solutions in Figs. (2.12)-(2.13) and their corresponding space-time metrics in Figs. (2.14)-(2.15). Here we have more input parameters (i.e., the number of dimensions, both in the bulk and on the brane) and free parameters (i.e., the values of...
gravitational coupling $\kappa$ and chiral coupling $\beta_N$). In this work we fix $p = 3$ since we are only interested in the most realistic case, the case where we live on 3-brane. For each number of co-dimension $N$ we have two free parameters, $\kappa$ and $\beta_N$.

For each $N$-codimension, we investigate the dependence of solutions on $\beta_N$. The behavior resembles its flat space-time counterparts. The coupling constant controls the thickness of defects. Here we only show solutions with $\beta_N = 1/2$ (strongly-coupled case). While their width is determined by $\beta_N$, their existence depends on $\kappa$. For a given $\beta_N$ the variation of $\kappa$ does not change its width, but (as in the case of the Skyrme branes) there exists a critical value, $\kappa_{\text{crit}}$, beyond which no solutions exist. The value of $\kappa_{\text{crit}}$ in each co-dimension $N$ depends also on $\beta_N$. For co-dimension $N = 4$ and $\beta_4 = 1/2$ we found that $\kappa_{\text{crit}} \sim 1/10$. Different $\beta_N$ (might) give different $\kappa_{\text{crit}}$. 

\[25d = 4 + 1\]
In this chiral-DBI branes we do not find any second branch of solutions, unlike the Skyrme branes case. This may be caused by the nature of DBI form; i.e., there is a maximum value of $K$ above which the Lagrangian becomes imaginary, thus non-physical. It seems that the \textit{square-root} form of the Lagrangian suppresses the existence of upper branch solutions.

![Typical forms of $B(r)$ and $H(r)$ profiles (with $\beta_4 = 1/2$) in $d = 4 + 1$. Both asymptote to flat space.](image)

The solutions are indeed the smooth and regular version of (2.53) and (2.54). In order to show this, we employ the same technique as before; i.e., we identify

\[
T_{\mu\nu}^{ADM} = \lim_{r \to \infty} \frac{1}{2K^2} \int d\Omega_{(N-1)} r^{(N-1)} \hat{r}^i \left[ \eta_{\mu\nu} \left( \partial_i h^\sigma_{\sigma} + \partial_i h^j_{\sigma} - \partial_j h^i_{\sigma} \right) - \partial_i h_{\mu\nu} \right],
\]

(2.60)

which, for our isotropic gauge, gives

\[
T_{\mu\nu}^{ADM} = \frac{\Omega_{(N-1)}}{2K^2} \lim_{r \to \infty} r^{(N-1)} \left[ pB(r)B'(r) + (N - 1)H(r)H'(r) \right] \eta_{\mu\nu},
\]

(2.61)
where $\Omega_{(N-1)}$ is the hypersurface area of an $(N-1)$-sphere. We can identify the single parameter $r_0$ that characterizes the p-brane solutions with its ADM mass for each given value of codimension. Plotting the vacuum and chiral-DBI branes solutions, figures (2.16) and (2.17), they asymptotically match at large radius.

\section*{2.4 Discussion}

In this chapter we study a mechanism to smooth out the naked singularity found in the (boost-symmetric) cosmic $p$-branes \cite{79}. Our proposal is that the singularity can be regularized by uncharged cosmic defect cores. We consider some generalizations of texture defects, in particular: Skyrmions and chiral DBI solitons. We construct the latter as stable higher co-dimensional extended defects that play similar role as Skyrme model in $4d$. The DBI-
Figure 2.16: Comparison between numerical and thin wall solutions \([2.53]\) of \(B(r)\) for 3-branes (with \(\beta_7 = 1/2\)) in \(d = 7 + 1\).

Figure 2.17: Comparison between numerical and thin wall solutions \([2.54]\) of \(H(r)\) for 3-branes (with \(\beta_7 = 1/2\)) in \(d = 7 + 1\).
form enables the theory to self-generate arbitrary higher-order terms that can evade Derrick’s theorem. The theory has one positive free-parameter, the DBI coupling constant \( \beta_2^2 N \). It is weakly-coupled (i.e., reduces to the ordinary texture defect) when \( \beta_2^2 N > 1 \), and becomes strongly-coupled when \( \beta_2^2 N < 1 \). We construct, using numerical techniques, explicit extended solitonic solutions up to \((8+1)\)-dimensions. Their sizes and asymptotic behavior are controlled by parameter \( \beta_2^2 N \). As \( \beta_2^2 N \) becomes greater than unity the defect width becomes thinner; as \( \beta_2^2 N \to \infty \) the size is infinitely thin, signaling the deformation of the defect configuration to the trivial vacuum. For small values of \( \beta_2^2 N \) the defect’s thickness becomes thicker and at the same time becomes more localized (i.e., the profile falls off faster asymptotically).

In \( 7d \) we obtain self-gravitating solitonic 3-brane solutions in the Einstein-Skyrme model, the Skyrme branes. The energy density is much localized and falls off fast enough asymptotically, regularizing the naked singularity of the vacuum solutions [79]. We found that for each value of gravitational coupling \( \hat{\kappa} \) there are two branches of solutions, the upper and lower branches. The latter is stable, possessing smaller tension, and correspond to the Skyrmions in flat space in the limit \( \hat{\kappa} \to 0 \). The upper branch corresponds to the pure Skyrme limit, and we conjecture to be unstable. The two branches merge at some critical value, which we numerically found to be \( \hat{\kappa}_{\text{crit}} \sim \frac{1}{20} \). Beyond this critical value there exists no static solution. We conjecture that the fate of super-critical configuration is either (i) collapsing to (Skyrme) black branes, or (ii) non-static (i.e., inflating) Skyrme branes.

In \((N+1)\)-dimensions we smooth out the \( p \)-branes singularity by constructing chiral-DBI branes. We show that there is no obstruction of obtain-
ing self-gravitating solitonic branes in this theory. The existence of solutions depends on the value of $\kappa$, while their size and asymptotic fall-off behavior are controlled by $\beta_N$. Unlike the case in the Skyrme branes, we found no branches of solutions. We conjecture that it is due to the DBI nature of the theory. The square-root form imposes an upper bound for the allowed physical solutions to exist. We also found critical values of $\kappa$ in this theory. The value of $\kappa_{\text{crit}}$ in general is dependent on the co-dimension of the theory and on the value of $\beta_N$. For example, in $N = 4$ and $\beta_4 = 1/2$ we found $\kappa_{\text{crit}} = 1/10$. Different number of $N$ and values of $\beta_N$ might give different $\kappa_{\text{crit}}$, and we have not studied this parameter space in detail in this dissertation.

All self-gravitating solitonic branes presented are asymptotically-flat, therefore represent a good candidate for regular braneworlds in the Dvali-Gabadadze-Porrati (DGP) \cite{DGP} model in $7d$. We attempted the DGP-gravity mechanism within our models but with no avail. We found no metastable graviton trapped on the brane. It is likely that additional fields are needed to realize such a mechanism (see refs. \cite{115, 116, 117, 118}). We do not pursue this possibility in this work.
Chapter 3

Decay of 6d Flux Vacua to Nothing

3.1 Flux Compactification in 6d Einstein-Maxwell Theory

Even though the necessity of extra dimensions gains a strong theoretical backing in the context of string theory, stabilizing their sizes has been one of the most challenging problems for building a realistic model of unification. At a phenomenological level various toy models of compactification have been constructed by invoking bulk fields \[119, 29, 31, 32, 33, 34, 120\]. These fields produce fluxes wrapping and winding the extra dimensions, stabilizing it against the internal-curvature-driven collapse. This mechanism is called flux compactification.

A simple toy model which remarkably is rich enough to illustrate many important features of more realistic flux compactifications of string theory \[121\].
is the 6d Einstein-Maxwell theory, proposed by Randjbar-Daemi et al [122] (also in [123]). In this scenario, compactification of two extra dimensions is achieved by having (abelian) magnetic fluxes threading the extra dimensions in such a way that the latter are compactified on an $S^2$ with definite size.

The action for this theory is given by

$$ S = \int d^6x \sqrt{-g} \left( \frac{1}{2\kappa^2} R - \frac{1}{4} F_{MN} F^{MN} - \Lambda \right). $$

(3.1)

The equations of motion are

$$ R_{MN} - \frac{1}{2} g_{MN} R = \kappa^2 T_{MN}, $$

$$ \frac{1}{\sqrt{-g}} \partial_M \left( \sqrt{-g} F^{MN} \right) = 0, $$

(3.2)

with energy-momentum tensor

$$ T_{MN} = g^{LP} F_{ML} F_{NP} - \frac{1}{4} g_{MN} F^2 - g_{MN} \Lambda. $$

(3.3)

We adopt the following ansatz

$$ ds^2 = g_{MN} dx^M dx^N = g_{\mu\nu} dx^\mu dx^\nu + C^2 d\Omega_2^2. $$

(3.4)

This ansatz spontaneously breaks the full Poincare symmetry into $X_4 \times S^2$, where $X_4$ is a 4d maximally-symmetric space-time endowed with the metric $g_{\mu\nu}$. Its Ricci scalar is given by $R^{(4)} = 12 H^2$, where $H^2$ can be positive (de Sitter), zero (Minkowski), or negative (anti-de Sitter). The compactification manifold is a 2-sphere of radius $C$. The 6d Einstein tensor is

$$ G_{\mu\nu} = -g_{\mu\nu} \left( 3H^2 + \frac{1}{C^2} \right), $$

$$ G_{ij} = -6H^2 g_{ij}, $$

(3.5)

1The 6d reduced Planck mass is written $M_{(6)} = 1/\sqrt{\kappa}$, and $\Lambda$ is the 6d cosmological constant.
where $\mu, \nu = 0, 1, 2, 3$ and $i, j = 4, 5$.

The field ansatz for the Maxwell sector is given by the monopole-type configuration \cite{122},

$$A_\varphi = -\frac{n}{2e} (\cos \theta \pm 1),$$

which is the only ansatz that respects the chosen isometries of the metric, and saturates the Dirac quantization condition \cite{112,122}, where $n \in \mathbb{Z}$ is the monopole number. We can read off the field strength tensor as

$$F_{\theta\phi} = -F_{\phi\theta} = \frac{n}{2e} \sin \theta,$$

that automatically satisfies the Maxwell’s equations. The corresponding energy-momentum tensor is given by

$$T_{\mu\nu} = -g_{\mu\nu} \left( \frac{n^2}{8e^2C^4} + \Lambda \right),$$

$$T_{ij} = g_{ij} \left( \frac{n^2}{8e^2C^4} - \Lambda \right).$$

Einstein’s equations lead to the relations for $H$ and $C$

$$3H^2 + \frac{1}{C^2} = \kappa^2 \left( \frac{n^2}{8e^2C^4} + \Lambda \right),$$

$$6H^2 = \kappa^2 \left( \Lambda - \frac{n^2}{8e^2C^4} \right).$$

This can be solved in terms of the parameters of the $6d$ theory and the magnetic flux number $n$, yielding the solutions

$$C^2 = \frac{1}{\kappa^2 \Lambda} \left( 1 \mp \sqrt{1 - \frac{3n^2}{4n_0^2}} \right),$$

$$H^2 = \frac{2\kappa^2 \Lambda}{9} \left[ 1 - \frac{2n_0^2}{3n^2} \left( 1 \pm \sqrt{1 - \frac{3n^2}{4n_0^2}} \right) \right],$$

\footnote{$2 \int_{S^2} F = 2\pi n/e$.}
where we have defined
\[ n_0^2 = \frac{2e^2}{\kappa^4 \Lambda}. \]  
(3.11)

We can see that by having sufficiently large flux \( n \), \( 1 \ll n \leq \frac{4}{3}n_0^2 \), we can switch from 4d anti-de Sitter \( AdS_4 \) to Minkowski \( M_4 \) or de Sitter \( dS_4 \) metrics while still realizing the compactifications of extra dimensions, \( C^2 > 0 \).

### 3.1.1 The Landscape of Vacua

To understand the richness of the spectrum of solutions, let us study this theory from a 4d perspective. We assume the following ansatz for the metric

\[ ds^2 = g_{MN} dx^M dx^N = e^{-\psi(x)/M_P} g^{(4)}_{\mu\nu} dx^\mu dx^\nu + e^{\psi(x)/M_P} C_0^2 d\Omega_2^2, \]  
(3.12)

with \( \psi \) the radion field characterizing the size of the extra dimensions and \( C_0 = 1/\sqrt{2\kappa^2 \Lambda} \). (The true size of extra dimensions is given by \( C = e^{\psi_{\text{min}}/(2M_P)} C_0 \), where \( \psi_{\text{min}} \) is the value of \( \psi \) that minimizes the potential.) Using this ansatz we Weyl-rescale and dimensionally-reduce the theory (see Appendix E), and arrive at the low-energy (effective) Action\(^3\)

\[ S = \int d^4x \sqrt{-g^{(4)}} \left( \frac{1}{2} M_P^2 R^{(4)} - \frac{1}{2} \partial_\mu \psi \partial^\mu \psi - V(\psi) \right), \]  
(3.13)

with the potential for the canonical radion \( \psi \) given by

\[ V(\psi) = \frac{4\pi}{\kappa^2} \left( \frac{n^2}{2n_0^2} e^{-3\psi/M_P} - e^{-2\psi/M_P} + \frac{1}{2} e^{-\psi/M_P} \right). \]  
(3.14)

The 4d Planck mass \( M_P \) is dependent on the volume of the compactification manifold via \( M_P^2 = 4\pi C^2 / \kappa^2 \).

\(^3\)The discussion here largely follows ref. \([124, 125]\).
The potential \( V(\psi) \) is depicted in Fig. (3.1) for \( n = 160, 180, 200, 220, 230 \), with \( n_0 = 200 \). We can see that for each value of flux number \( n \) there are two stationary points, one of which is stable (minimum) under small perturbation and the other one, once perturbed, will either roll down to the stable vacuum or decompactify (\( i.e., \) roll down indefinitely to \( \psi \to \infty \)). The two solutions in (3.10) correspond to these two stationary points, with the stable one corresponds to the upper-sign solution. This is the solution we are mostly interested in. Notice that by increasing \( n \) we lift up the minima of the potential, and so there exists a critical value of \( n \), \( n_{\text{crit}} \), beyond which the minima disappear and the field \( \psi \) will roll down unboundedly to infinity. These collections of minima of the potential constitute a landscape of vacua.

Figure 3.1: The 4d effective radion potential \( V(\psi) \) (in reduced Planck units) as a function of the radion field \( \psi \). Plotted are the shapes of the potential for five different flux values \( n = 160, 180, 200, 220, 230 \).
characterized by the flux number $n$, similar to (but less complicated than) the string theory landscape. For $n < n_0$ the value of $H^2$ is negative, i.e., AdS space-time. For $n = n_0$ the 4d space is Minkowski (flat), while for $n > n_0$ the positive-value of the minima corresponds to $H^2 > 0$, i.e., 4d de Sitter. We have three general types of vacua here: $AdS_4 \times S^2$, $M_4 \times S^2$, and $dS_4 \times S^2$. For 4d AdS and Minkowski their vacua are stable with respect to decompactification, since each of their vacua are global minimum of the potential. However, for 4d de Sitter their vacua are unstable against decompactification to $dS_6$ via barrier penetration; the size of extra dimensions destabilizes and grows with time. The tunneling probability gets higher as the flux number increases. For very-high $dS_4$ vacua even classical perturbation can render it unstable.

3.1.2 Tunnelings in the Landscape

None of the vacua (for $n \neq 0$) are absolutely stable; they are all unstable against decay. By studying the landscape of the potential, there are at least two decay channels: (i) tunneling between different flux vacua $n$, and (ii) (de)compactifications to (and/or from) $dS_6$. Tunnelings between them have been studied in [124, 125, 128]. We need to find instantons that interpolate between the two vacua. To tunnel to vacua with lower flux number we have to introduce an object which removes flux. Since extra dimensions are threaded with magnetic flux, we need magnetically charged extended objects (e.g., branes) to unravel them, analogous to the nucleation of monopole-antimonopole pairs in a homogeneous magnetic field [129]. Flux tunneling are

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4 The AdS vacua are terminal; they all end at big crunch singularity.
5 The possibility of decay of $dS_6$ into $dS_4 \times S^2$ or $dS_2 \times R^4$ (from “true” vacuum to “false” vacuum) was pointed out in refs. [126, 127].
mediated by instantons contructed from the magnetically charged (extremal) black 2-branes [124] which can be found in the spectrum of the theory [130, 79].

On the other hand, it has recently been suggested in [71] that a simple toy model of axionic flux compactification can have “exotic” transitions where, instead of tunneling to the nearest vacua, the decays occur through tunneling all the way to \( n = 0 \). Here, the brane acts as a sink for all the flux and renders the space-time collapse in a non-singular way, creating a large coordinate region of volume measure zero. This is a bubble of nothing in flux compactification. The surface of the bubble is a de Sitter vortex charged with respect to the axion. Due to the simplicity of the model considered, the only allowed vacuum is \( AdS_4 \times S^1 \).

This raises a question whether it can happen also in this 6d flux compactification, which has richer vacua; i.e., that the 6d Einstein-Maxwell theory admits bubbles of nothing in its spectrum of decay channels. The natural choice for the bubble wall in 6d theory is a magnetically-charged brane. Since the wall is non-singular, we need a solution where the extra-dimensional space-time is smooth everywhere. This is difficult to achieve in the Abelian model, since it seems to inevitably lead to a singularity at the location of the monopole. Here topological defects (once again) lend a hand. We resolve the singularity by embedding the model in a non-abelian gauge theory which is known to possess smooth magnetically charged solitons of co-dimension three (i.e., ’t Hooft-Polyakov monopoles), the Yang-Mills-Higgs model [17, 18]. We shall show that this landscape does indeed suffer from non-perturbative instability of tunneling to nothing [131].
3.2 The Einstein-Yang-Mills-Higgs Landscape

To study (smooth) bubbles of nothing in 6d Einstein-Maxwell flux compactification we can imagine embedding the Einstein-Maxwell theory into more complicated models which include new degrees of freedom that modify the UV regime only, and so would not distort the landscape of 4d flux vacua computed previously. Here we realize this with a specific Einstein-Yang-Mills-Higgs \( SU(2) \) model. This is one of the first flux compactification models described in the literature \[119\]. Let us study the landscape in this theory.

The model is defined by the action

\[
S = \int d^6x \sqrt{-g} \left( \frac{1}{2\kappa^2} R - \frac{1}{4} F_{aMN} F^{aMN} - \frac{1}{2} D_M \Phi^a D^M \Phi^a - V(\Phi) - \Lambda \right),
\]  

(3.15)

with

\[
V(\Phi) = \frac{\lambda}{4} (\Phi^a \Phi^a - \eta^2)^2,
\]

\[
F_{aMN} = \partial_M A^a_N - \partial_N A^a_M + e\epsilon^{abc} A^b_M A^c_N,
\]  

(3.16)

\[
D_M \Phi^a = \partial_M \Phi^a + e\epsilon^{abc} A^b_M \Phi^c.
\]

Varying the action with respect to the fields yields the equations of motion

\[
R_{AB} - \frac{1}{2} g_{AB} R = \kappa^2 T_{AB},
\]  

(3.17)

\[
\frac{1}{\sqrt{-g}} D_M \left( \sqrt{-g} D^M \Phi \right)^a = \lambda \Phi^a (\Phi^b \Phi^b - \eta^2)^2,
\]  

(3.18)

\[
\frac{1}{\sqrt{-g}} D_N \left( \sqrt{-g} F^{MN} \right)^a = e\epsilon^{abc} (D^M \Phi^b) \Phi^c,
\]  

(3.19)

where the energy-momentum tensor is given by

\[
T_{AB} = D_A \Phi^a D_B \Phi^a + F_{aAM} F^{aM} + g_{AB} \mathcal{L},
\]  

(3.20)
with

$$\mathcal{L} = - \frac{1}{2} D_a \Phi^a D^a \Phi^a - \frac{1}{4} \mathcal{F}_{MN}^{a} \mathcal{F}^{aMN} - V(\Phi) - \Lambda. \quad (3.21)$$

Cremmer and Scherk [119] showed that compactification solutions of the 6d space-time exist in the spectrum of the theory. They restricted themselves to the flat 4d space-time $\mathbb{R}^{1,3} \times S^2$ and for $n = 1$. They obtained several types of solutions. In Appendix F we discuss in detail some of the peculiar properties of this type of compactification which are special to $n = 1$. Here we generalize such compactifications to arbitrary integer $n$ flux vacua by choosing a matter field ansatz as follows

$$\Phi^a = \eta \, p_c (\sin \theta \cos n\varphi, \sin \theta \sin n\varphi, \cos \theta),$$

$$A^a_\mu = A^a_r = 0,$$

$$A^a_\theta = \frac{1 - w_c}{e} (\sin n\varphi, - \cos n\varphi, 0),$$

$$A^a_\varphi = \frac{n(1 - w_c)}{e} \sin \theta (\cos \theta \cos n\varphi, \cos \theta \sin n\varphi, - \sin \theta), \quad (3.22)$$

with $n \in \mathbb{Z}$. The suitability of this ansatz can be motivated by computing the topological charge for this configuration [132] via

$$\frac{1}{4\pi} \int d\theta d\phi |\Phi|^{-3} \varepsilon_{abc} \Phi^a \partial_\theta \Phi^b \partial_\phi \Phi^c = n, \quad (3.23)$$

with $|\Phi| = \sqrt{\Phi^a \Phi^a}$. For $p_c = 1$ and $w_c = 0$ the ansatz reduces to the Maxwell case. These values corresponds to the far-field boundary conditions, where they asymptote to the Abelian flux vacua. This can be understood by looking at the form of the electromagnetic tensor, eq.(1.17), which in this case becomes

$$F_{\theta \phi} = \frac{n}{e} \sin \theta. \quad (3.24)$$
The equations of motion reduce to

\[ 3H^2 + \frac{1}{C^2} = \kappa^2 \left( \frac{n^2}{2e^2C^4} + \Lambda \right), \]
\[ 6H^2 = \kappa^2 \left( \frac{n^2}{2e^2C^4} \right). \]

(3.25)

Note that the difference between Dirac quantization (eq.(1.12)) and Schwinger (eq.(1.22)) quantization conditions manifests itself in the discrepancy of definition of the coupling constant $e$ with respect to the abelian case. The charge $e$ here is twice the same charge in Maxwell theory. This is expected since Schwinger’s condition is always twice the Dirac’s.

As in the case of Einstein-Maxwell landscape, we only consider perturbatively stable solutions of the equations of motion. Following the arguments presented in the Einstein-Maxwell theory, these are specified by the two length scales

\[ C^2 = \frac{1}{\kappa^2\Lambda} \left( 1 \mp \sqrt{1 - \frac{3\kappa^4\Lambda n^2}{2e^2}} \right), \]
\[ H^2 = \frac{2\kappa^2\Lambda}{9} \left[ 1 - \frac{e^2}{3\Lambda\kappa^4n^2} \left( 1 \pm \sqrt{1 - \frac{3\kappa^4\Lambda n^2}{2e^2}} \right) \right]. \]

(3.26)

The landscape of vacua is identical to the pure electromagnetic case (since we set $p_c = 1$ and $w_c = 0$), in particular we see that the theory has $4d$ compactifications $AdS_4 \times S^2$, $\mathbb{R}^{1,3} \times S^2$, and $dS_4 \times S^2$. 

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3.3 Bubbles of Nothing Solutions

Unfortunately, the ansatz (3.22) is not suitable for studying bubble of nothing in the general landscape. This is due to a no-go theorem that states that there exists no spherically-symmetric (i.e., hedgehog) monopole solutions with $n > 1$ \cite{[133, 134]}. It implies that ansatz (3.22) is valid only for describing bubble of nothing in $n = 1$.

In \cite{[71]} bubbles of nothing in 5d axionic flux compactification were discussed. The extra dimension is compactified on a circle. The corresponding instantons are constructed from vortex defects charged with phase winding number, whose world-volume is a co-dimension two de Sitter object. In this work, we want to find a similar object in higher dimensional space-time where the compactification manifold is a 2-sphere. The appropriate ansatz then is

$$ds^2 = B(r)(-dt^2 + \cosh^2 t \, d\Omega_2^2) + dr^2 + C^2(r)(d\theta^2 + \sin^2 \theta \, d\varphi^2). \quad (3.27)$$

As in the case of Witten’s bubble of nothing, in the limit of $r \to 0$ the $r$-slice degenerates into a 2 + 1 dimensional de Sitter space $dS_3$, the wall-induced metric. At large $r$ the ansatz approaches the flux compactification vacua provided we impose $C(\infty) = C_\infty = constant$ and appropriate boundary conditions for $B(r)$, depending on whether the corresponding vacua is $AdS_4 \times S^2$, $M_4 \times S^2$, or $dS_4 \times S^2$.

We are looking for solutions that describe the decay of flux compactifications to a bubble of nothing, i.e., solutions where the extra-dimensional space wound with magnetic flux degenerates to a point at some value of $r$, which we choose to be at $r = 0$. This implies the existence of a magnetic source at the degeneration loci, which we satisfy by placing a solitonic magnetic
brane centered at \( r = 0 \), making use of our UV completion of the low energy Einstein-Maxwell theory. An appropriate ansatz is therefore the *hedgehog* configuration,

\[
\Phi^a = \eta p(r) (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta),
\]

\[
A^a_{\mu} = A^a_r = 0,
\]

\[
A^a_{\theta} = \frac{1 - w(r)}{e} (\sin \varphi, - \cos \varphi, 0),
\]

\[
A^a_{\varphi} = \frac{1 - w(r)}{e} \sin \theta (\cos \theta \cos \varphi, \cos \theta \sin \varphi, - \sin \theta),
\]

(3.28)

using which the Yang-Mills-Higgs equations, (3.18)-(3.19), reduce to

\[
p'' + \left( 3 \frac{B'}{B} + 2 \frac{C'}{C} \right) p' - \frac{2w^2p}{C^2} - \lambda \eta^2 p (p^2 - 1) = 0,
\]

(3.29)

and

\[
w'' + 3 \frac{B'}{B} w' + \frac{w(1 - w^2)}{C^2} - \epsilon^2 \eta^2 p^2 w = 0,
\]

(3.30)

respectively.

The Einstein equations from the metric (3.27) are

\[
G^0_0 = - \frac{1}{B^2} - 1 \frac{C'}{C^2} + \left( \frac{B'}{B} \right)^2 + 4 \frac{B'C'}{BC} + \left( \frac{C'}{C} \right)^2 + 2 \frac{B''}{B} + 2 \frac{C''}{C} = \kappa^2 T^0_0,
\]

(3.31)

\[
G^r_r = - \frac{3}{B^2} - 1 \frac{C'}{C^2} + 3 \left( \frac{B'}{B} \right)^2 + 6 \frac{B'C'}{BC} + \left( \frac{C'}{C} \right)^2 = \kappa^2 T^r_r,
\]

\[
G^\theta_\theta = - \frac{3}{B^2} + 3 \left( \frac{B'}{B} \right)^2 + 3 \frac{B'C'}{BC} + 3 \frac{B''}{B} + \frac{C''}{C} = \kappa^2 T^\theta_\theta,
\]

with energy-momentum tensor specified by

\[
T^0_0 = - \left[ \eta^2 \left( \frac{p^2}{2} + \frac{p^2 w^2}{C^2} \right) + \frac{1}{\epsilon^2 C^2} \left( w^2 + \frac{(1 - w^2)^2}{2C^2} \right) + \frac{\lambda \eta^4}{4} (p^2 - 1)^2 + \Lambda \right],
\]
\( T_r = \eta^2 \left( \frac{p'^2}{2} - \frac{p'^2 w'^2}{C^2} \right) + \frac{1}{\epsilon^2 C^2} \left( w'^2 - \frac{(1 - w^2)^2}{2C^2} \right) - \frac{\lambda \eta^4}{4} (p^2 - 1)^2 - \Lambda, \)
\( T_\theta = -\frac{\eta^2 p'^2}{2} + \frac{(1 - w^2)^2}{2 \epsilon^2 C^4} - \frac{\lambda \eta^4}{4} (p^2 - 1)^2 - \Lambda, \)
\( T_\phi = T_\theta. \)

(3.32)

Eqs. (3.29)-(3.31) define the system of equations we need to solve to obtain the bubbles of nothing. As can be seen from Fig. (3.1) there are three possibilities of 4d vacua from which the bubbles nucleate: AdS\(_4\), M\(_4\), and dS\(_4\). They are specified by fixing the values of \( e \) and \( \Lambda \). We will study the existence of bubbles of nothing in each of those three possible asymptotic 4d effective geometries. We found, as we shall see, qualitatively different solutions depending on the values of the other two fundamental parameters of the 6d theory, \( \eta \) and \( \lambda \).

To avoid singularity at the core (i.e., the bubble wall) we require the solutions behave like power-series at \( r \to 0 \),
\[
\begin{align*}
   p(r) &= p_1 r + \cdots, \\
   w(r) &= 1 + w_2 r^2 + \cdots, \\
   B(r) &= B_0 + B_2 r^2 + \cdots, \\
   C(r) &= r + C_3 r^3 + \cdots.
\end{align*}
\]

Using the equations of motion we can write all coefficients, \( B_2, C_3 \), etc., in terms of three locally undetermined constants, \( B_0, p_1, \) and \( w_2 \). They are

\[
p(r) = p_1 r + \cdots.
\]

\(^6\)Note that these type of equations have been discussed and solved in the context of \textit{inflating magnetically-charged branes} in [135]. In this dissertation we consider them with different interpretation.
\[
\begin{align*}
    w(r) &= 1 + w_2r^2 + \cdots \\
    B(r) &= B_0 + B_0 \left[ \frac{1}{12B_0^2} + \frac{3(2e^2 + B_0^2 e^2 \eta^2 p_2^2 \kappa^2 + 8B_0^2 w_2^2 \kappa^2)}{2B_0^2 e^2} \
    &\quad \quad \quad \quad \quad - \frac{\kappa^2}{4} \left( 6\eta^2 p_1^2 + \eta^4 \lambda + \frac{24w_2^2}{e^2} + 4\Lambda \right) \right] r^2 + \cdots, \\
    C(r) &= r - \frac{(2e^2 + B_0^2 e^2 \eta^2 p_2^2 \kappa^2 + 8B_0^2 w_2^2 \kappa^2)}{12B_0^2 e^2} r^3 + \cdots.
\end{align*}
\]

(3.34)

To obtain numerical solutions we once again employ the shooting method with \(B_0, p_1,\) and \(w_2\) as shooting parameters. However, unlike in the case of Skyrmie and chiral DBI branes, the method seems to require very-high precision fine tuning when applied to this problem, thus it becomes very difficult to find smooth non-singular solutions. This is because we have more fields to solve: two metric fields \((B(r)\) and \(C(r)\)) and two matter fields \((p(r)\) and \(w(r)\)). So instead, we use the modification technique of shooting method: multiple shooting [136] (also used in [3]). In this method, instead of shooting from the core all the way to the asymptotic boundaries, we divide the range of integrations into several slices of regions. For each interval we evaluate the numerical integrations. By having smaller range, the numerical method converges faster than single shooting. Besides having to satisfy the core and asymptotic conditions, the acceptable solutions obviously must match and be smooth at the junctions. Therefore we impose additional conditions, that the functions must be continuous and differentiable across the boundaries.

### 3.3.1 Bubbles of Nothing in \(AdS_4 \times S^2\)

The first compactification we consider is to \(AdS_4 \times S^2\), which occurs for all values of \(n\) in a landscape with \(\Lambda \leq 0\), or for \(n < e^2 / (2\kappa^4 \Lambda)\) with any value
of $\Lambda$. In [71] the existence of bubbles of nothing were studied, where the landscape of vacua are all of this type. The minimal case is $\Lambda = 0$, i.e., the Freund-Rubin type of flux compactification [123]. In order to construct the

![Figure 3.2: A bubble of nothing in a Freund-Rubin-type of AdS compactification. The bubble wall is at $r = 0$, where the 2-sphere degenerates. The “warp factor” $B(r)$ is non-zero at the wall, and goes exponentially toward the AdS boundary, where all the fields relax to their vacua. Throughout, we use reduced Planck units ($\kappa = 1$).](image)

bubble of nothing for this case, we impose boundary conditions compatible with the asymptotic compactification geometry. Within our $SO(1, 3) \times SO(3)$ invariant metric ansatz Eq. (3.27), the asymptotically $AdS_4 \times S^2$ solution is

$$p(r) \to 1, \quad w(r) \to 0,$$
\[ C(r) \to C_\infty, \quad B'(r)/B(r) \to |H|, \]

\[ (3.35) \]

as \( r \to \infty \). The values of \( |H| \) and \( C_\infty \) for the \( n = 1, \Lambda = 0 \) case can be seen in eq.(3.26) to be

\[ C_\infty = \sqrt{\frac{3\kappa^2}{4e^2}}, \]

\[ |H| = \sqrt{\frac{4e^2}{27\kappa^2}}. \]

\[ (3.36) \]

The full solutions (Figs.(3.2)-(3.3)) interpolate smoothly between the core, whose expansion is given by eq.(3.33), and the asymptotic solution, eq.(3.35).

Figure 3.3: A typical bubble of nothing in non-minimal (\( \Lambda < 0 \)) AdS compactification.
From 6d point of view, the behavior of warp factor $B(r)$ can be interpreted as the presence of a throat-like region in our space-time. Since $B(r)$ is decreasing as we move closer toward the core, the bubble wall is gravitationally attractive. In the literature, gravitationally attractive throats appear in the context of warped compactifications [137]. A gapped warped throat (e.g., Klebanov-Strassler) in global coordinates may even be perceived of as a bubble of nothing geometry, albeit with cylindrical rather than de Sitter isometry, and lacking a negative mode. In our solutions, the bubble wall represents the smooth termination of this throat. We can use the gravitational properties of the throat as a proxy for the effective 4d tension of the bubble. Since a bubble of nothing accelerates toward an outside observer, the throat is attractive, and the apparent 4d tension of the domain wall is negative [72, 138].

It is not, in some sense, really instructive to discuss the existence of decay to nothing in AdS vacua, since all AdS spaces are bound to collapse in a big crunch singularity. Nevertheless, our results show that $AdS_4 \times S^2$ space is burdened with non-perturbative instability, i.e., a non-singular tunneling to nothing.

### 3.3.2 Bubbles of Nothing in $M_4 \times S^2$

We can uplift the effective 4d cosmological constant to zero for the $n = 1$ vacua by raising the 6d cosmological constant to

$$
\Lambda = \frac{e^2}{2\kappa^4},
$$

such that the vacuum is now $M_4 \times S^2$. We show that decay to nothing exists
Figure 3.4: A bubble of nothing in Minkowski compactification.

as well in this type of vacua. The asymptotic solution is then given by

\[
\begin{align*}
    p(r) & \to 1, & w(r) & \to 0, \\
    C(r) & \to C_\infty = \frac{\kappa}{\epsilon}, & B'(r) & \to 1,
\end{align*}
\]

(3.38)

as \( r \to \infty \). A numerical solution for \( \mathbb{M}_4 \times S^2 \) instability to nothing is depicted in fig. (3.4).

As mentioned before, there exists a region in parameter space \( \lambda - \eta \) where the solutions asymptote to the compactification condition but the behavior at the wall is slightly different. This is shown in fig. (3.5). There, the warp factor \( B(r) \) displays a “punt” shape, like a wine bottle achieving its minimum slightly away from the core. We believe that this variation of the warp factor is due to the competition between the two contributions to the 4d gravitational
Figure 3.5: A variation of bubble of nothing solution in Minkowski compactification. The minimum of the “warp factor” $B(r)$ does not occur at the bubble wall.
properties in this region, one coming from the bubble of nothing itself, and
the other from the magnetic 2-brane located at the surface of the bubble.

### 3.3.3 Bubbles of Nothing in $dS_4 \times S^2$

Unlike in the previous two cases, bubbles of nothing in de Sitter space is com-
plicated by the existence of a cosmological horizon. The bubble’s exterior
region has finite radius, $0 \leq r \leq r_h$, where the cosmological horizon radius $r_h$
is such that $B(r_h) = 0$.

Expanded about the horizon at $r = r_h$, the solution takes the form

$$
\begin{align*}
  p(r) &= p_h + p_2 (r - r_h)^2 + \cdots, \\
  w(r) &= w_h + w_2 (r - r_h)^2 \cdots, \\
  B(r) &= (r - r_h) - B_3 (r - r_h)^3 + \cdots, \\
  C(r) &= C_h - C_2 (r - r_h)^2 + \cdots.
\end{align*}
$$

(3.39)

where $p_h, w_h, C_h$ are free parameters,\footnote{They become the “shooting parameters” at the horizon} in terms of which all other coefficients of this expansion are completely determined, given by

$$
\begin{align*}
p_h^2 &= \frac{p_h [2w_h^2 - \lambda \eta^2 C_h^2 (1 - p_h)]}{8C_h^2}, \\
w_h^2 &= \frac{w_h [w_h (w_h + e^2 \eta^2 p_h^2 C_h^2) - 1]}{8C_h^2}, \\
B_3 &= \frac{1}{288C_h^4 e^2} \left\{ 10(w_h^2 - 1)^2 \kappa^2 + 8e^2 C_h^2 (\eta^2 p_h^2 w_h^2 \kappa^2 - 1) \\
  &\quad - e^2 C_h^4 \kappa^2 \left[ \eta^4 \lambda (p_h^2 - 1) + 4\Lambda \right] \right\}, \\
C_2 &= \frac{1}{48C_h^3 e^2} \left\{ 2\kappa^2 (w_h^2 - 1)^2 + 4e^2 C_h^2 (\kappa^2 w_h^2 \eta^2 p_h^2 - 1) \right\}.
\end{align*}
$$

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\[ +e^2C_h^4(\kappa^2(\eta^4\lambda(p_h^2 - 1)^2 + 4\Lambda)) + \frac{1}{4e^2C_h^4}
\left[10\kappa^2(w_h^2 - 1)^2 + 4\Lambda\right] \]
\[ + 8e^2C_h^2(\eta^2p_h^2w_h^2\kappa^2 - 1) - e^2C_h^4\kappa^2(\eta^4\lambda(p_h^2 - 1)^2 + 4\Lambda)) \rceil. \]

(3.40)

(3.41)

One typical solution is shown in Fig.(3.6). The solutions are only defined up

to \( r = r_h \).

So far we classify bubble of nothing as boost-invariant solutions which
degenerate smoothly at the core. For the AdS and Minkowski cases, this
definition appears to be sufficient. However, for de Sitter space, it no longer
is. Bubbles of nothing (with metric (3.27)) in $dS_4 \times S^2$ are topologically equivalent to $dS_6$ defects. To be precise, consider the anisotropic slicing of $dS_6$, whose metric is given by $B(r) = \cos r, C(r) = \sin r$:

$$ds^2 = \cos^2 r \left( -dt^2 + \cosh^2 t \, d\Omega_2^2 \right) + dr^2 + \sin^2 r d\Omega_2^2. \quad (3.42)$$

Remarkably, this appears to be a bubble of nothing! In this case, any observer is on the core of the bubble at $r = 0$ and sees a cosmological horizon at $r = \pi/2$.

This example forces us to adopt a more restrictive definition of bubble of nothing so that it can be distinguished from any other physical solutions. We therefore require that the solutions also asymptote to $4d$ flux vacuum\(^8\). This condition implies that the acceptable solutions should asymptotically be stable, \textit{e.g.}, the radius of extra dimensions remain compactified and do not grow with time. This is accomplished by determining our solutions beyond the cosmological horizon, which we denote by region II in Fig.\((3.7)\), and see its evolution with time.

To do this, we analytically continue our metric (3.27) across this horizon via the substitution $r \rightarrow it$ and $t \rightarrow \chi + i\pi/2$, yielding

$$ds^2 = -dt^2 + B^2(t) d\mathcal{H}_3^2 + C^2(t) d\Omega_2^2, \quad (3.43)$$

where $d\mathcal{H}_3^2$ is the unit metric on three-dimensional hyperbolic space,

$$d\mathcal{H}_3^2 = d\chi^2 + \sinh^2 \chi d\Omega_2^2. \quad (3.44)$$

The time-dependent Einstein’s equations are then given by

$$G_0^0 = \frac{3}{B(t)^2} - \frac{1}{C(t)^2} - 3 \left( \frac{B'(t)}{B(t)} \right)^2 - 6 \frac{B'(t)C'(t)}{B(t)C(t)} - \left( \frac{C'(t)}{C(t)} \right)^2 = \kappa^2 T_0^0, \quad (3.45)$$

\(^8\)All bubbles of nothing solutions in anti-de Sitter and Minkowski satisfy this condition, as can easily be seen from Figs.\((3.2)-(3.5)\).
Figure 3.7: A conformal diagram of a bubble of nothing in $dS^4$. The spacetime is only defined in the shaded region.

\[
G^r_r = \frac{1}{B(t)^2} - \frac{1}{C(t)^2} - \left( \frac{B'(t)}{B(t)} \right)^2 - 4 \frac{B'(t)C'(t)}{B(t)C(t)} - \left( \frac{C'(t)}{C(t)} \right)^2,
\]
\[
-2 \frac{B''(t)}{B(t)} - 2 \frac{C''(t)}{C(t)} = \kappa^2 T^r_r,
\]
\[
G^\theta_\theta = \frac{3}{B(t)^2} - 3 \left( \frac{B'(t)}{B(t)} \right)^2 - 3 \frac{B'(t)C'(t)}{B(t)C(t)} - 3 \frac{B''(t)}{B(t)} - \frac{C''(t)}{C(t)} = \kappa^2 T^\theta_\theta.
\]

(3.45)

Note that in this time-like equation the zeroth-zeroth and $rr$-components of the Einstein tensors switch role, compared to eq. (3.31).

For the matter fields in this region, we adopt ansatz as follows

\[
\Phi^a(t) = \eta p(t)(\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta),
\]
\[
A^a_\mu(t) = A^a_r(t) = 0,
\]
\[
A^a_\theta(t) = \frac{1 - w(t)}{e} (\sin \varphi, - \cos \varphi, 0),
\]
\[
A^a_\varphi(t) = \frac{1 - w(t)}{e} \sin \theta (\cos \theta \cos \varphi, \cos \theta \sin \varphi, - \sin \theta).
\]

(3.46)
We ensure the suitability of this ansatz by checking that it satisfies the Bianchi identity

\[ D_\mu \tilde{F}^{\mu \nu} = 0. \] (3.47)

The corresponding time-dependent energy-momentum tensors are

\[
T^0_0 = -\left[ \eta^2 \left( \frac{p'(t)^2}{2} + \frac{p(t)^2 w(t)^2}{C(t)^2} \right) + \frac{1}{e^2 C(t)^2} \left( w'(t)^2 + \frac{(1 - w(t)^2)^2}{2C(t)^2} \right) \right. \\
+ \frac{\lambda \eta^4}{4} (p(t)^2 - 1)^2 + \Lambda \left], \\
T^r_r = \eta^2 \left( \frac{p'(t)^2}{2} - \frac{p(t)^2 w(t)^2}{C(t)^2} \right) + \frac{1}{e^2 C(t)^2} \left( w'(t)^2 - \frac{(1 - w(t)^2)^2}{2C(t)^2} \right) \\
- \frac{\lambda \eta^4}{4} (p(t)^2 - 1)^2 - \Lambda, \\
T^\theta_\theta = \eta^2 \frac{p'(t)^2}{2} + \frac{(1 - w(t)^2)^2}{2e^2 C(t)^4} - \frac{\lambda \eta^4}{4} (p(t)^2 - 1)^2 - \Lambda,
\] (3.48)

while the time-dependent Higgs and Yang-Mills equations, eqs. (3.29) and (3.30), are, respectively,

\[
p''(t) + \left( \frac{3B'(t)}{B(t)} + 2 \frac{C'(t)}{C(t)} \right) p'(t) + \frac{2w(t)^2 p(t)}{C(t)^2} + \lambda \eta^2 p(t)(p(t)^2 - 1) = 0 \quad (3.49)
\]

and

\[
w''(t) + \frac{3B'(t)}{B(t)} w'(t) - \frac{w(t)(1 - w(t)^2)}{C(t)^2} + e^2 \eta^2 p(t)^2 w(t) = 0. \quad (3.50)
\]

The general expansion of the fields about the light-cone \((t = 0)\) yields

\[
p(t) = p_h + p_2 t^2 + \cdots, \\
w(t) = w_h + w_2 t^2 + \cdots, \\
B(t) = t + B_3 t^3 + \cdots, \\
C(t) = C_h + C_2 t^2 + \cdots,
\] (3.51)
Figure 3.8: The evolution of the scale factor $B(t)$ and radion field $C(t)$ beyond the cosmological horizon for the bubble of nothing solution shown in Fig. (3.6). Scale factor $B(t)$ grows exponentially while the size of extra dimensions $C(t)$ remains compactified around $C_h$, indicating the stability of the asymptotic region, $dS_4 \times S^2$.

Figure 3.9: The evolution of the scalar $p(t)$ and vector $w(t)$ fields beyond the cosmological horizon for the bubble of nothing solution shown in Fig. (3.6). At late time they all relax to their vacuum manifold, indicating the stability of flux compactification in the asymptotic region.
where the three undetermined coefficients $p_h, w_h$, and $C_h$ are trivially related to the field values across the horizon ($r = r_h$). Eqs. (3.45)-(3.50) govern the time-evolution of the asymptotic compactification of the bubbles of nothing solutions. For the typical solution shown in Fig. (3.6) we can see that at late time they all relax to the 4d $dS_4 \times S^2$ compactification. This is shown in Figs. (3.8)-(3.9).

There is however, a different class of solution one can find in this future directed region. These are solutions which lead to runaway behavior for the radion $C(t)$, leading to decompactification. It is not difficult, in fact, to see that at late cosmological time this geometry asymptotes to 6d de Sitter space written in an anisotropic gauge. They should not be interpreted as a bubble of nothing in a flux compactification, but instead as an instanton describing the creation of smooth magnetically charged 2-branes in $dS_6$ [125].

Another distinct family of solutions we obtained is that the ones with purely-repulsive bubble wall, i.e., the warp factor $B(r)$ is a monotonically-decreasing function of the distance from the wall. We found that they do not suit the interpretation of bubbles of nothing either. We believe that they are more appropriately perceived as the spontaneous creation of an open flux compactification; a bubble from nothing. We shall elaborate this in more detail in the next chapter.

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9 We shall not discuss this class of solution here. See appendix E of ref. [131].
3.4 Discussion

Bubbles of nothing are terminal vacuum. They are the last destination any space-time can decay into (besides, of course, big crunch singularity). We have studied this phenomenon within the framework of $6d$ flux compactification, where the extra dimensions are stabilized by magnetic flux. We found that the bubble wall should necessarily be magnetically-charged, and we accomplished it by having magnetically-charged solitonic brane, found in the spectrum of $6d$ Einstein-Yang-Mills-Higgs theory. Our solutions generalize the previously-found solutions in [71], to include the decay of all possible vacua in $6d$ Einstein-Maxwell theory [122, 123].

The difference between our solutions with Witten’s bubble of nothing [64] or the ones in axionic flux compactification [71] are

- The solutions describe a smooth degeneration of $S^2$, rather than the previously known $S^1$, corresponding to the number of compactified dimensions.

- Since the size of extra dimensions is stabilized by magnetic flux, the bubble wall is magnetically-charged.

- The bubbles can have either positive or negative $4d$ effective tensions, parametrized by the values of parameters $\eta$ and $\lambda$.

- The spherically-symmetric solutions preserve the isometry of the compactification manifold only for flux number $n = \pm 1$. In this work, we only study $n = 1 \rightarrow n = 0$ transition. However, we find no conceptual obstruction in studying decay from higher flux number $n$, but they will
not obey hedgehog ansatz, eq. (1.14).

Bubbles of nothing in flux compactification open up a new decay channel of landscape vacua, some of which were thought to be stable and model our $4d$ world. Surely these are relevant for the study of measure problems in the context of multiverse landscape (see, for example, ref. [139]).
Chapter 4

Creation of Open Flux Universe from Nothing

Among our bubble of nothing solutions in $dS_4 \times S^2$ in the previous chapter there is a family of them with peculiar behavior: they have horizon at a finite distance from the core, and the warp factor $B(r)$ is an ever-decreasing function of distance; i.e., the bubble wall is repulsive. All observers are thrown away from the bubble. We do not find the interpretation as bubbles of nothing fit here. We are therefore led to interpret them differently. They describe a “reverse-process” of space-time decay to nothing. A nucleation of a space-time from nothing.

This interpretation brings a profound implications since it enters the realm of quantum cosmology, where one of its hot topic is to address the problem of the beginning. We will review briefly below how science (i.e., cosmology) attempts to answer such a metaphysical question.
4.1 Creation of Universe from Nothing

One of the greatest question in cosmology is: *how (and why) did the universe begin?* For centuries this question has been pondered (mostly) by theologians and philosophers. With the progress of cosmology for the last hundred years, however, it now also becomes a question of physics.

It started with the Einstein’s theory of general relativity. Not long after it was published, Friedman (and independently Lemaitre) employed the equations to show that the universe is generically expanding and, if we trace back to the past, there was a moment where space and time were singular, the big bang. The theory soon gained wide acceptance as it was supported by abundant observational results (*e.g.*, abundance of deuterium, cosmic background radiation, the expansion of universe, *etc*). One of its implication is that the universe must have a finite age, cannot be infinitely old. The big bang was regarded as the moment the universe was born. What happened before big bang is unknown. In fact, the big bang theory is actually the theory of the aftermath of the big bang itself, since space-time is singular and physics breaks down at that point.

Soon after the theory of inflationary universe was proposed (to remedy several unnatural conditions in the big bang theory, *e.g.*, horizon problem) [140, 141, 142] it was realized that inflation is generically eternal [143, 144]. The universe keeps undergoing inflationary stage; it may stop in some local regions but globally the inflation keeps going on and never stops. If it is eternal to the future, it is very tempting to imagine that it can also be eternal to the past. One can imagine that our universe is just one among infinitely many other
universes stretched out by an inflation which happened infinitely many times in the past as will keep happening infinitely many times in the future. If this picture is correct, then we can avoid the question of the beginning. The universe was not born at one particular time in the past, but rather simply existed and will continue to exist. It is infinitely old. Unfortunately this idea turns out to be incorrect. Under very general assumptions, Borde et al \cite{145} proves a theorem that an inflationary universe is geodesically incomplete to the past; the universe cannot be past-eternal. There should still be the beginning.

Vilenkin \cite{146, 147, 148} suggested a possibility that the universe might have tunneled to a de Sitter space from nothing. The tunneling is mediated by a compact instanton, and its analytic continuation to Lorentzian signature describes the subsequent evolution of a closed de Sitter universe. The instanton is given by

\[
ds^2 = d\sigma^2 + a^2(\sigma)\left(d\psi^2 + \sin^2\psi d\Omega_2^2\right),
\]

with \(a(\sigma) = H^{-1} \cos(H\sigma)\) and \(H = \sqrt{\frac{8\pi G V(\phi_0)}{3}}\), where \(V(\phi)\) is a scalar field potential. This instanton describes a compact manifold, an \(S^4\) with radius \(H^{-1}\). This can be interpreted as tunneling from nothing. Upon analytic continuation \(\sigma \to \text{it}\), the Lorentzian metric describes a closed Friedman-Robertson-Walker universe. From this model’s point of view, the beginning is Nothingness.

Vilenkin’s model of creation of universe assumes a closed universe, since it is difficult to see how a spatially infinite universe could emerge from a compact instanton. On the other hand, the instanton describing the nucleation of a bubble universe from a false vacuum (i.e., Coleman-De Luccia instanton \cite{67}) shows us how to provide an open universe out of the tunneling. The symme-
try of the decay process is such that the interior of the bubble can be shown to be an open universe \[67, 149\]. Extending this idea, we can imagine constructing an instanton mediating decay of an open universe but without any false vacuum. This will be a representation of a creation of an open universe from nothing. This type of instanton has been proposed by Hawking and Turok \[150\] to mediate a tunneling of true vacuum (of an open space-time) without any false vacuum. Their instanton has only an $O(4)$ invariance, a more general ansatz than that of Vilenkin’s, with Euclidean metric

$$ds^2 = d\sigma^2 + b^2(\sigma) \left( d\psi^2 + \sin^2 \psi d\Omega_2^2 \right), \quad (4.2)$$

where $b(\sigma)$ is the radius of the 3-sphere, which in turn is the “great circle” of the $S^4$. Upon analytic continuation $\psi = \pi/2 + i\tau$, the Lorentzian metric becomes

$$ds^2 = d\sigma^2 + b^2(\sigma) \left( -d\tau^2 + \cosh^2 \tau d\Omega_2^2 \right), \quad (4.3)$$

a spatially-inhomogeneous de Sitter-like metric. This metric describes the exterior of the inflating bubble \[150\]. Continuation through the null surface ($\sigma = 0$), $\sigma \rightarrow it$ and $\tau \rightarrow i\pi/2 + \xi$, yields

$$ds^2 = -dt^2 + a^2(t) \left( d\xi^2 + \sinh^2 \xi d\Omega_2^2 \right), \quad (4.4)$$

with $a(t) \equiv ib(\sigma)$. This metric describes an expanding open FRW universe.

There is a price to pay, however. This Hawking-Turok instanton is singular. It is precisely this singularity that enables the decay without any false vacuum. This singularity, nevertheless, is mild enough so that they can proceed to integrate the Euclidean Action and obtain a finite result. However, the validity of this instanton is judged with some criticism since it would imply that a flat universe could decay by a similar process \[151\].

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One solution to this singularity problem is proposed by Garriga [72]. He considered a five-dimensional extension of the Hawking-Turok theory with Action as follows:

\[ S_E = \frac{1}{16\pi G_5} \int d^5x \sqrt{g} R. \]  

(4.5)

His ansatz for the Euclidean metric is

\[ ds^2 = d\tau^2 + R^2(\tau)dS_3 + r^2(\tau)dy^2, \]

(4.6)

possessing an \( O(4) \times O(1) \)-symmetry. Solving the Einstein’s equations, the solution is smooth and regular,

\[ ds^2 = d\tau^2 + (\tau^2 + A^2)dS_3 + \left( \frac{\tau^2}{\tau^2 + A^2} \right) dy^2, \]

(4.7)

with \( A \) a positive integration constant. Upon closer inspection this instanton describes bubble of nothing in 5d Kaluza-Klein vacuum, as in [64]. Moreover, under Weyl-rescaling

\[ g_{AB} = e^{2\kappa^2\phi/3} \begin{pmatrix} 0 & 0 \\ 0 & e^{-2\kappa^2\phi} \end{pmatrix}, \]

(4.8)

and dimensional reduction, the Action (4.5) reduces to

\[ S = \int \sqrt{-g} \left[ - \frac{\mathcal{R}}{2\kappa^2} + \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi \right], \]

(4.9)

the Action for Hawking-Turok instanton. Therefore, this singularity is a mere artifact of dimensional reduction to 4d universe. This is because Witten’s bubble of nothing is smooth and non-singular. In this theory, the flat space-time is still metastable against decay, but the decay rate can be made exponentially small if the size of the extra dimension is much larger than the Planck scale [72].

\footnote{\textsuperscript{1}This Action should more appropriately be supplemented with the Hawking’s boundary term \( S_H \propto \int d^4\xi \sqrt{-\kappa}, \) where \( \kappa \) is the extrinsic curvature.}
4.2 Bubble from Nothing

Here we are considering the creation of an open 6d universe stabilized via flux compactification from nothing, which shares many of the characteristics of the Garriga’s resolutions \[152\]^2. The existence of (two) extra dimensions will allow us to obtain a compact, smooth solution of the higher dimensional equations of motion that would be singular looking from a 4d point of view.

Figure 4.1: A typical $AdS_4 \times S^2$ bubble from nothing for $\Lambda = \frac{2\kappa^2}{3\kappa^4}$.

Our theory is still described by Lagrangian (3.1), with the Euclidean metric ansatz

$$ds^2 = B^2(r) \left( d\psi^2 + \sin^2 \psi d\Omega_2^2 \right) + dr^2 + C^2(r)(d\theta^2 + \sin^2 \theta d\varphi^2). \quad (4.10)$$

\[2\] For a similar proposal in the context of string theory, albeit with non-open universe, see \[153\].
Figure 4.2: A minimal (Freund-Rubin) $AdS_4 \times S^2$ bubble from nothing ($\Lambda = 0$).

We impose the following boundary conditions

\[ B(0) = B_0, \quad B(r_h) = 0, \]
\[ C(0) = 0, \quad C(r_h) = C_h. \quad (4.11) \]

In the previous chapter, similar boundary conditions gave rise to bubbles of nothing solutions in $dS_4 \times S^2$. Here, we shall show that the same boundary conditions for different values of parameters $\eta$ and $\lambda$ can produce spontaneous creation of $AdS_4 \times S^2$, $M_4 \times S^2$, and $dS_4 \times S^2$ spaces from nothing.

As before, we will only work within the spherically-symmetric ansatz (3.28), which requires us to stay in $n = 1$. Since these solutions are topologically equivalent to the $dS$-bubbles of nothing, we use the same expansions of (3.33) around the core and (3.39) at the horizon. The 4d conformal diagram of this bubbles from nothing resembles the diagram for the bubbles of nothing,
Fig. (1.2), with the difference is that space-time is only defined outside the shaded region. After the nucleation the bubble will undergo its evolution depending on which vacua it tunnels to (i.e., $AdS_4 \times S^2$, $M_4 \times S^2$, or $dS \times S^2$).

For the nucleation of a bubble in $AdS_4 \times S^2$ space, we found solutions for both minimal (Freund-Rubin) and non-minimal ($\Lambda \neq 0$) cases. They are shown in Fig. (4.1)-(4.2).

Figure 4.3: An $M_4 \times S^2$ bubble from nothing.

In $M_4 \times S^2$ vacuum we are able to obtain solutions for a large range of $\eta-\lambda$ parameter space. One typical solution is shown in Fig. (4.3). The bubble from nothing with the least tension we found is with $\eta = 1.486$ and $\lambda = 6.7275$.

One could also find a family of solution within the Minkowski compactification where the scalar and vector fields do not completely relax to their vacua. This is shown in Fig. (4.4). We shall see later that upon analytic contin-
Figure 4.4: One typical solution of $M_4 \times S^2$ bubble from nothing that can give rise to post-nucleation inflation.

Figure 4.5: A $dS_4 \times S^2$ bubble from nothing.
uation this type of family of solutions can have an interesting behavior which resembles the very early era of our universe after its nucleation.

There exist bubble from nothing solutions in $dS_4 \times S^2$ compactification as well. One typical de Sitter bubble from nothing solution is shown in Fig.4.5.

### 4.3 The Lorentzian Continuation: Open Inflation

We show the nucleation solutions of three types of possible vacua in $6d$ Einstein-Maxwell flux compactification from nothing. The metric at the nucleation “moment” is given by the Euclidean form, eq.(4.10). We can analytically-continue the instanton solution back into the Lorentzian signature, by $\psi \rightarrow \frac{\pi}{2} + it$:

$$ds^2 = B^2(r) \left( -dt^2 + \cosh^2 t d\Omega_2^2 \right) + dr^2 + C^2(r) \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right). \quad (4.12)$$

This, together with the expansion around the core makes it clear that the geometry has an inflating magnetically-charged object sitting at its end, i.e., a co-dimension three de Sitter brane. Cosmological horizon exists at $r = r_h$ where $B(r_h) = 0$ and the size of the extra dimensional manifold is fixed. This situation is very similar to what happens in the space-time geometry of a global string [154], except that in our case the region beyond the horizon has compactified $S^2$ extra dimensions stabilized by the magnetic flux.

To describe the evolution of the bubble after its nucleation, we should investigate its behavior beyond the horizon. To do this, we use the same prescription as we did for the case of bubble of nothing in de Sitter space;
that is, we extend the geometry beyond the horizon as given by eq. (3.43) and adopt the time-dependence for all the fields, then we evolve them according to the time-dependent Einstein’s, Higgs’, and Yang-Mills’ equations. To describe pure compactification the radion field $C(t)$ should be stabilized at its compactification value, as should the scalar and vector fields. The interesting point is that one can tune the parameters of the theory (i.e., $\eta$ and $\lambda$) such that the subsequent evolution of this open universe ends up in the basin of attraction of the flux compactified vacua; i.e., that we can adjust the values of $\eta$ and $\lambda$ so that the bubbles evolve into our compactification regime. Obviously there are some numerical solutions obtained which do not obey this future-behavior; i.e., their radion grows with time and so the extra dimensions decompactify.

Figure 4.6: The time-evolution of the scale factor $B(t)$ and the radion $C(t)$. Here $B(t)$ behaves exponentially at early time before turning power-law at late time. The radion $C(t)$ acts as the inflaton, staying at the top of its potential (represented by the constant line, which is the lower-sign/unstable solution of eq. (3.26)) for quite a period of time before rolls down to its stable compactification. This is obtained for $\eta = 2.4070426$ and $\lambda = 6.7275$. 

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An interesting result happens when we consider the Minkowski-type compactification. There exists some values of $\eta$ and $\lambda$ where the late-time evolution of the $4d$ part stabilizes at matter-dominated regime (due to the oscillation of $C(t)$), i.e., the scale factor $B(t) \sim t^{2/3}$, but at early time goes exponentially, $B(t) \sim e^t$. This behavior is shown in Fig. (4.6), where we evolve the bubble

![Figure 4.7: The time-evolution of the scalar $p(t)$ and vector fields $w(t)$. Both relax to the compactified value at late time. This is obtained for $\eta = 2.4070426$ and $\lambda = 6.7275$.](image)

from nothing solution in Fig. (4.4). The early-time behavior shows exponential inflation with $e$-foldings $\sim 17$, before going power-law. Inflation is driven by the radion $C(t)$ which plays the role of inflaton, staying at the top of its potential driving exponential expansion, before falling back to its stable compactification value. Like all bubble interiors, the universe begins curvature-dominated ($B(t)$ grows linearly) and is rapidly followed by vacuum-energy-dominated epoch, which eventually gives way to a long matter-dominated epoch. Finally, curvature will dominate again at times much later than shown in the figure.
4.4 Smooth Transition: \textit{Bubble Of Nothing} $\rightarrow$ \textit{Bubble From Nothing}

Spontaneous creation from nothing and non-perturbative decay to nothing exist within a same toy model of compactification, the 6\textit{d} Einstein-Maxwell theory. The interior of bubble \textit{from} nothing resembles an inside-out version of the bubble \textit{of} nothing solutions studied in the previous chapter. This might suggest that a universe created from a bubble from nothing (BFN) could decay by a bubble of nothing (BON). We found that this is \textit{not} the case. In fact, what we found is that there is a smooth transition in the $\eta$-$\lambda$ parameter space from bubble of nothing to bubble from nothing.

Recall that the choice of vacua depends on the value of $e$ and $\Lambda$, and within the same vacuum energy sector the tension of the bubble is determined by $\eta$ and $\lambda$. By exploring the parameter space we can have different behaviors of the bubbles. We observed that by increasing the tension the numerical solutions interpolate smoothly from bubble of nothing to bubble from nothing. This signals the stability of the universe created by a bubble from nothing; \textit{i.e.}, that a bubble from nothing cannot decay into a bubble of nothing. Obviously a rigorous proof of this stability would be to fully explore the whole parameter space for each vacua and find no BFN and BON solutions that overlap within the same values of $\eta$ and $\lambda$. Here, we take a modest route. We probe only along the line $\lambda = 6.7275$ of $M_4 \times S^2$, and found that each type of bubble solution is uniquely determined by $\eta$ and $\lambda$. 
4.4.1 Bubbles of Nothing \((1 < \eta < \eta_{\text{crit}})\)

We studied this region in the previous chapter. The bubble wall is gravitationally attractive. A typical solution with small values of \(\eta \gtrsim 1\) is shown in Fig.\((3.4)\).

As we increase \(\eta\), the warp factor \(B(r)\) develops wrinkles at its core; the minimum does not occur at the bubble wall. We dubbed it “punted bubble”. Physically this implies a short-range repulsive gravity near the bubble wall, while still preserving its long-range attractive behavior far away from the core. One typical solution is shown in Fig.\((3.5)\).

4.4.2 The Critical Bubble \((\eta = \eta_{\text{crit}})\)

By keeping increasing \(\eta\) the \(dS_3\) curvature of the bubble wall gets smaller. At \(\eta = \eta_{\text{crit}}\) the curvature is zero, i.e., the bubble wall is flat. This flat brane is the boundary between BON and BFN walls. We can regard this solution as a static non-inflating magnetic flat brane. We found that \(\eta_{\text{crit}} = 1.447928125\). The solution is shown in Fig.\((4.8)\).

4.4.3 Bubbles from Nothing \((\eta_{\text{crit}} < \eta < \eta_d)\)

Further increasing \(\eta\) results in another inflating magnetically-charged solitonic brane, but having a cosmological horizon at some distance \(r = r_h\) away from the core. This is none other the bubble from nothing we are studying in this chapter. One typical \(M_4 \times S^2\) bubble from nothing with small tension is depicted in Fig.\((4.3)\).

\(^3\text{Note that our model does not admit stable compactifications for } \eta < 1.\)
Figure 4.8: The critical bubble in $M_4 \times S^2$ for $\lambda = 6.7275$. The value of the warp factor at the core, $B(0)$, is very large. The induced metric on the wall is very close to a flat brane and not inflating. Shown here is the warp factor $B(r)$. All other field solutions are not depicted clearly as they are very small compared to $B(r)$.
As explained in the previous sections, looking at these solutions beyond the horizon reveals a stable compactification. Most of them, however, do not mimic our very-early history of universe, \textit{i.e.,} they do not possess inflationary behavior. Increasing $\eta$ changes the value of the matter fields at the horizon. At $\eta = 2.4070426$ the scalar and vector fields do not fully relax to their corresponding vacua. This configuration somehow propels the radion field outward to the top of its effective compactification potential for a period of time, during which it in turns drives an exponential expansion of the scale factor. This is the solution shown in Fig.\ref{fig:4.4}, where its beyond-horizon evolution is described in Fig.\ref{fig:4.6}-\ref{fig:4.7}. After several e-folds the radion rolls back to its stable vacuum, as do the scalar and vector fields.

\subsection{4.4.4 Decompactification ($\eta > \eta_d$)}

The value of $\eta$ that gives rise to inflationary behavior marks the boundary between bubble from nothing and decompactification. In this decompactification regime, the scale factor $B(t)$ keeps exponentially expanding, does not enter the matter-dominated era. This is caused by the radion field $C(t)$ which, instead of rolling down to the stable compactification, rolls to infinity. As a result, the extra dimensions decompactify.

Finally, there exists a limit of $\eta$ beyond which the values of the matter fields do not change from those at the core of the defect, \textit{i.e.,} $p(r) = 0$, $w(r) = 1$. We can think of these solutions as magnetically-charged branes whose cores are larger than the radius of $dS_6$. The warp factor $B(r)$ and the radion field $C(r)$ behave like $\cos(r)$ and $\sin(r)$, respectively. The metric then is given by the anisotropic slicing of $dS_6$, eq.\ref{eq:3.32}. This is nothing more than \textit{topological}
Figure 4.9: Topological inflation in the 6\textit{d} Einstein-Maxwell flux compactification. This can be regarded as a pure \(dS_6\) decompactification.

\textit{inflation} \[94, 95\] in the 6\textit{d} theory. We found that the parameter value that just falls into this regime is given by \(\eta = 2.57\). This is shown in Fig. (4.9).

\section*{4.5 Discussion}

We have identified and studied 6\textit{d} instantons that describe the creation \textit{from} nothing of an open 4\textit{d} universe in the context of flux compactification. The higher-dimensional nature of the solutions eliminates the singularities found in the 4\textit{d} instantons of this type \[150\]. The brane core allows for a smooth degeneration of the extra-dimensional space, while at the same time acts as a magnetic flux source that stabilizes, in the asymptotic regime, the modulus associated with the extra dimensions.
After nucleation, according to the standard big bang cosmology and the inflationary paradigm the universe must enter the stage of inflation and ends with reheating. Most of the BFN solutions obtained do not describe this behavior. Nevertheless, we found a neighborhood in parameter space which gives a post-nucleation inflationary universe with \( \sim 17 \) e-folds. Obviously any realistic model of inflation should ensure the existence of a period of slow-roll sufficient to have no less than \( \sim 60 \) e-folds within the bubble. We have been unable to fine-tune the \( \eta - \lambda \) parameter space to obtain sufficiently large e-folds to mimic our early universe. One should also ensure that the spectrum of perturbations is consistent with observations. This requirement might necessarily need new ingredients to be added into the theory. We do not attempt to pursue this possibility. Nevertheless, our work shows that within a rather simple toy model we are able to show how an open flux universe can be spontaneously created from nothing and undergoes evolution describes by standard cosmological model afterwards.

We also investigated the stability of BFN. In particular, we investigated whether BFN is semi-classically unstable and decay to BON. We found that for a particular vacua, \( M_4 \times S^2 \), and along the line \( \lambda = 6.7275 \) there is no overlap of BFN with BON. Instead, we obtained a smooth transition from BON to BFN and all the way to the pure \( dS_6 \) decompactification (\( i.e., \) topological inflation) as we increase the value of \( \eta \). In addition, we found a flat bubble as the boundary of transition. This critical bubble can be regarded as a flat “domain wall” end to an \( M_4 \times S^2 \) compactification. Surely we do not claim that we give a rigorous proof of uniqueness of BON and BFN. Rather, we only conjecture that based on this investigation the BFN seems to be stable.
against decay to nothing by nucleating BON and that the boundary between
the two is marked with a critical bubble, even though we suspect that the
transition in de Sitter compactifications is less sharp due to the impossibility
of Minkowski domain walls in dS space.
Chapter 5

Conclusions

We have studied the role of topological defects in extra-dimensional cosmology. This dissertation can be divided into two parts, depending on the type of defects considered and the asymptotic topology of the corresponding extra dimensions.

5.1 Texture Defects in Braneworld Scenario

This topic is covered in chapter 2. Here we explored a mechanism to regularize the naked singularity found in the boost-symmetric $p$-branes in [79] using uncharged solitonic defects having finite total energy. We considered defects with non-trivial third homotopy group $\pi_3(\mathcal{M}) = \mathbb{Z}$, i.e., texture branes. In order to be stable in static condition, it is necessary for textures in $N \geq 3$ spatial dimensions to have higher-order kinetic terms. We accomplished this by considering two types of non-canonical textures: the Skyrme model and the chiral-DBI soliton.

We studied the $7d$ Einstein-Skyrme theory and obtained self-gravitating...
solitonic 3-brane solutions in its spectrum, numerically. The energy-momentum
tensor of the Skyrme field is able to smooth out the naked singularity at
the core, while at the same time is localized enough that asymptotically ap-
proaches the (thin-wall) $p$-brane solutions [79]. The solutions are parametrized
by a free parameter $\hat{\kappa} \equiv eF_0\kappa$, where $e$ and $F_0$ are parameters found in the
the Skyrme model and $\kappa$ is the gravitational coupling. Upon obtaining the
numerical solutions we encountered the fact that there is a branch of two
solutions for each value of $\hat{\kappa}$, the lower branch and the upper branch. The
lower branch is stable, having smaller tension, and at the limit of $\hat{\kappa} \to 0$ cor-
responds to the decoupling of the Einstein-Skyrme system, approaching the
flat Skyrmion solutions. The upper branch, on the other hand, is conjectured
to be unstable. The instability of its 4d counterpart has been studied and
confirmed in [86]. Another way of looking at this instability is to recall that
at the limit of $\hat{\kappa} \to 0$, either $\kappa \to 0$ or $F_0 \to 0$. The latter corresponds to the lower branch (decoupling of Skyrme model from gravity), while the former
corresponds to the upper branch. At $F_0 \to 0$ the first term of the Skyrme
Lagrangian, eq.(1.36) becomes sub-dominant. This is the pure Skyrme limit,
discussed in [100]. By Derrick’s argument [14] this system does not possess
any stable static solution, and in fact it was suggested in [86] and shown
in [104] that the Einstein-Skyrme model limit reduces to the Einstein-Yang-
Mills model. This theory has metastable solutions, the Bartnik-McKinnon
solitons [101].

In this 7d Einstein-Skyrme theory there exists a critical coupling constant
$\hat{\kappa}_{\text{crit}}$ at which the two branches of solutions merge, and beyond which there
is no static solution. There are two possibilities of the fate of the superheavy
Skyrme branes. First, they might collapse forming a Skyrme black-brane. Second, they are inflating. We found that the critical value is given by $\tilde{\kappa}_{\text{crit}} \sim 1/20$.

In $N > 3$ co-dimensions the Skyrme model is no longer able to support stable static solutions. Here we proposed chiral-DBI soliton, first discussed in [113], to regularize naked singularity in arbitrary higher-dimensional $p$-branes. We showed that Derrick’s theorem does not forbid the theory to have stable static solutions in co-dimensions higher-than three. We constructed flat extended chiral-DBI solitons up to co-dimension\(^\dag\) 8. The solutions are parametrized by a single free parameter, $\beta_N$, where $N$ is the number of co-dimension of the brane. In order to have physical solutions, $\beta_N$ should be positive, $\beta_N > 0$. For large value of $\beta_N$, the theory is weakly-coupled, and upon Taylor expansion reduces to the usual non-linear sigma model, possessing (ordinary) texture defects in its spectrum of solutions. By Derrick’s theorem such a solution can be continuously deformed into the trivial vacuum. This behavior is shown in Fig. (2.10), where, as $\beta_N$ increases the defect thickness becomes smaller. At the limit of $\beta_N \to \infty$ the defect is infinitely thin, signaling the non-existence of natural scale. On the other hand for $0 < \beta_N < 1$ the theory becomes strongly-coupled. Taylor expansion breaks down here, and to obtain solutions we have to consider the full non-perturbative nature of the theory. We can see from Fig. (2.10) that for smaller $\beta_N$ the defects become thicker; the natural scale increases. The thickness also depends on the co-dimensions considered. As $N$ increases, so does the thickness. This is shown in Fig. (2.11).

\(^\dag\)In principle there is no obstruction to obtain solutions in any higher co-dimensions.
We then proceed to solve the Einstein-chiral-DBI equations and obtained self-gravitating solitonic chiral-DBI branes. The solutions are additionally parametrized by $\kappa$, the gravitational coupling. We solved the equations numerically for several co-dimensions. Unlike the case of Skyrme branes, there is no branch of solutions here. This is due to the DBI-nature of the theory, putting an upper bound for the allowed gradient energy. We also found that $\kappa_{\text{crit}}$ is a function of $\beta_N$ for a given co-dimension $N$. For co-dimension 4 solution with $\beta_4 = 1/2$, we found that the critical coupling constant is given by $\kappa \sim 1/10$.

All solutions considered in chapter 2 are asymptotically-flat, i.e., possessing an $M_d$ asymptotic topology. They therefore represent good candidates for realizing the Dvali-Gabadadze-Porrati (DGP) gravity [63]. We tried it but unsuccessful. It is likely that more ingredients are needed into the theory, see [116, 115, 117, 118].

5.2 Monopoles in Flux Compactification

The second part of this dissertation discusses semi-classical transitions in a toy model of flux compactification. We considered a simple model which shares many properties reminiscent to the realistic string theory flux compactification [121], the 6d Einstein-Maxwell theory [122, 123]. In this theory, the extra dimensions are compactified on $S^2$, stabilized by magnetic flux. By invoking 6d cosmological constant, we can have 4d de Sitter ($dS_4 \times S^2$), Minkowski ($M_4 \times S^2$), and anti-de Sitter ($AdS_4 \times S^2$) as its possible vacua. The landscape of vacua is characterized by flux number $n$, and the transition channels
between them (or to/from $dS_6$ decompactification) have been greatly studied in [124, 125, 128].

On the other hand, there is a transition that has not been discussed so far; transition all the way to $n = 0$. This is a bubble of nothing [64] in flux compactification. In this phenomenon, the extra dimensions degenerates to a point, and the space-time pinches off in a non-singular way, rendering it disappear to nothing. The bubble wall is non-singular, and acts as a magnetic source to preserve the flux. This is accomplished by having solitonic magnetic brane, exists within the spectrum of solutions of the Einstein-Yang-Mills-Higgs theory. In chapter 3 we studied this decay channel, and found that all vacua are unstable against this decay, *i.e.*, nucleating a bubble with no (classical) space-time inside. For the $M_4 \times S^2$ vacuum, increasing the bubble tension develops a wrinkle on the bubble wall. The minimum of the warp factor $B(r)$ occurs at some distance away from the core. Physically it implies the existence of a short-range repulsive gravity while still preserves the long-range attractive behavior. We dubbed these solutions “punted bubbles of nothing”.

Unlike in $AdS_4 \times S^2$ and $M_4 \times S^2$, the bubble of nothing in $dS_4 \times S^2$ is complicated by the existence of cosmological horizon. The anisotropic slicing of $dS_6$ appears to be the same as the bubble of nothing. Therefore, we investigated the evolution of the bubble beyond its horizon to distinguish it from $dS_6$ slicing or inflating branes. Upon analytic continuation the instanton describes an open 4d universe and compact extra dimensions on $S^2$. The time-evolution beyond the horizon must describe this compactification behavior, *i.e.*, the scale factor must grow exponentially, the radion is compactified, and the matter fields relax to their corresponding vacua. Most of the solutions
show this behavior. As we increase the parameters $\eta$ and $\lambda$, however, their time evolution fails to compactify. In this case, the radion rolls to infinity and the extra dimensions, in turn, decompactify.

Upon studying the behavior of bubbles of nothing beyond its horizon in $dS_4 \times S^2$ we found solutions where the bubble wall is gravitationally repulsive. The warp factor $B(r)$ is a monotonically-decreasing function of $r$. All observers outside are thrown away from the bubble. We found that the definition of bubble of nothing fits here. We are therefore led to interpret them as "the reverse-process" of decay to nothing: *nucleation of open universe from nothing*. We dubbed it "bubble from nothing", and studied it in chapter 4. This instanton can be seen as the higher-dimensional version of Hawking-Turok (HT) instanton [150] that mediates tunneling of an open universe from nothing. One important difference is that our instanton is regular. As has been pointed out by Garriga [72], the singularity in HT instanton can be regarded as a mere artifact of dimensional reduction of an otherwise regular higher-dimensional instanton. We investigated the spontaneous nucleation of all the vacua in the 6d theory.

Studying the time evolution of the $M_4 \times S^2$ bubbles from nothing, we found a neighborhood in parameter space that gives rise to the inflationary behavior. We could get inflation up to $\sim 17$ e-folds with $\eta = 2.4070426$. This is clearly insufficient to model our real universe which requires no less than $\sim 60$ e-folds. Nevertheless, it opens up a possibility that within a rather simple toy model we can have an open flux universe spontaneously created from nothing, and undergoes evolution describes by standard cosmological model afterwards.

We conjecture that the universe created from nothing cannot decay back
into nothing, i.e., that there is no region in parameter space where bubble from nothing (BFN) overlaps with bubble of nothing (BON). We investigated the line $\lambda = 6.7275$ for different value of $\eta$ in $M_4 \times S^2$. We found, as we increased $\eta$, a smooth transition, starting from BON to BFN and all the way to decompactification. By continuity, there must be a flat bubble, marking the boundary between BON and BFN. We found this flat bubble at $\eta = 1.447928125$. The boundary between BFN and decompactification is the one that gives the greatest e-folds, $\eta = 2.4070426$. Keep increasing $\eta$ results in the extreme limit of decompactification, the one where all the matter fields stay at their corresponding values at the core everywhere. This is a pure $dS_6$ decompactification, and can be regarded as topological inflation $^{94}$ $^{95}$. 
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Appendix A

Some Basic Homotopy Theory

Topology is a branch of mathematics that studies continuity \[155\] \[156\] \[157\]. In topology, geometrical shapes do not matter. Topology does not “care” with size, shape, distance, or angles; it deals only with continuous transformation. Stretching, squeezing, or twisting does not change the topology; tearing, cutting, or gluing does.

Continuous transformations are characterized by the concept of homeomorphism \[158\]. Let \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) be smooth manifolds. They are called homeomorphic to one another if there exists a continuous mapping \( \Phi : \mathcal{M}_1 \to \mathcal{M}_2 \) where the inverse is also continuous. If two or more manifolds are homeomorphic then they are said to belong to the same topological classes. The task of topology then is to fully characterize all equivalence classes defined from homeomorphisms and to place the manifolds in their appropriate classes \[158\].

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A.1 Homotopy Theory

Consider a smooth manifold $\mathcal{M}$. Let $x$ be a point on $\mathcal{M}$, $x \in \mathcal{M}$. A loop $\sigma$ at $x$ is any path which starts and ends at $x$. This can be defined in terms of continuous mapping $\Psi$ from the interval $0 \leq t \leq 1$, $t \in \mathcal{L}$, into $\mathcal{M}$; $\Psi : \mathcal{L} \to \mathcal{M}$, requiring that $\Psi(0) = \Psi(1) = x$. Two loops $\sigma$ and $\sigma'$ are called homotopic at $x$ if $\sigma$ can be continuously deformed into $\sigma'$ while still keeping contact with $x$. Homotopic is an equivalence relation:

- it is symmetric, since time flow can be reversed; i.e., $\sigma$ homotopic to $\sigma'$ implies $\sigma'$ homotopic to $\sigma$.
- it is transitive, since time intervals can be adjoined and rescaled; i.e., $\sigma$ homotopic to $\sigma'$ and $\sigma'$ homotopic to $\sigma''$ implies $\sigma$ homotopic to $\sigma''$.
- it is reflexive; i.e., $\sigma$ homotopic to itself.

All possible loops at $x$ can be subdivided into distinct homotopy classes, and under class multiplication they form a group.

Manifold $\mathcal{M}$ is called simply connected if, for any $x$, any two or more loops passing through $x$ are homotopic, or, equivalently, if every loop is homotopic to a point. Otherwise, the manifold is said to be non-simply connected or multi-connected. Simply-connected spaces fall into constant homotopy class. Euclidean spaces $\mathbb{R}^n$, $n \geq 1$, and the spheres $S^n$, $n \geq 2$, are examples of simply-connected manifolds. On the other hand, circle $S^1$, cylinder $S^1 \times \mathbb{R}$, and torus $S^1 \times S^1$ are non-simply connected. The homotopy group that classifies it is called the first homotopy group or the fundamental group of the manifold $\mathcal{M}$ at $x$, $\pi_1(\mathcal{M})$. 
For simply-connected manifold, $\pi_1(\mathcal{M}) = \mathbb{I}$, where $\mathbb{I}$ is the trivial group with only identity element. Therefore we can conclude that $\pi_1(S^n) = \mathbb{I}$, with $n \geq 2$. Non-simply connected manifolds have non-trivial first homotopy group. They are characterized by a group of integer, $\mathbb{Z}$, i.e., $\pi_1(S^1) = \mathbb{Z}$. This can be interpreted as how many times we can wrap a loop onto a hole in a non-simply-connected manifold. For this reason, the integer number $k \in \mathbb{Z}$ is called winding number.

In higher dimensional manifold the set of homotopy classes is denoted by $\pi_n(\mathcal{M})$. They also form a group, the $n^{th}$-homotopy group of $\mathcal{M}$. These homotopy groups define equivalence classes from $S^n$ into $\mathcal{M}$. In $n \geq 2$-dimensional manifold we can define the notion of contractibility. A manifold is said to be contractible if any closed surface on it is homotopic to a point. Clearly $\mathbb{R}^n$ is contractible. The simplest non-contractible would be $S^2$, a 2-sphere. This is characterized by homotopy group $\pi_2$, $\pi_2(S^2) = \mathbb{Z}$. In general, calculations for homotopy groups $\pi_n(\mathcal{M})$ is a very non-trivial task of algebraic topology, and much are still open problems. Some of the known results are listed below:

- $\pi_n(S^1) = \mathbb{I}$.
- $\pi_n(S^n) = \mathbb{Z}$.
- $\pi_3(S^2) = \mathbb{Z} \footnote{This result is highly non-trivial and counter-intuitive since previously it was thought that $\pi_n(S^n)$ is always trivial for every $n \neq m$. This counter-proof was found by Heinz Hopf, and this mapping is called Hopf Fibration.}$
A.2 Fundamental Theorems of Homotopy Theory

Consider a simply-connected continuous Lie group $G$ under which the field $\phi$ is invariant. The system is minimized when $\phi$ is in the ground states (vacuum manifold). There will exist a subgroup $H$ of $G$, whose elements leave the vacuum invariant. Therefore the vacuum manifold is equivalent (isomorphic) to the coset space $\mathcal{M} = G/H$. Let us now state (without proof) two fundamental theorems which will be our topological tools to classify defects:

- **First Fundamental Theorem**
  
  Let $G$ be a connected and simply-connected Lie group (its zeroth and first homotopy group are trivial), and let $H$ be a subgroup of $G$. Then:
  \[ \pi_1(G/H) \cong \pi_0(H). \]  
  \( \text{(A.1)} \)

- **Second Fundamental Theorem**
  
  Let $G$ be a connected and simply-connected Lie group, and let $H$ be a subgroup of $G$. Then:
  \[ \pi_2(G/H) \cong \pi_1(H). \]  
  \( \text{(A.2)} \)

\[^{2}\text{It can be a trivial group.}\]
\[^{3}\text{$\pi_0(\mathcal{M})$ is the homotopy classes of maps from a point ($S^0$) into $\mathcal{M}$. It characterizes the disconnectedness of manifold $\mathcal{M}$.}\]
Appendix B

Derrick’s Theorem

Consider a theory of \( n \) scalar fields \( \phi^a, a = 1, 2, \ldots, n \) in \((d + 1)\)-dimensional space-time \([13, 11, 12]\) with positive-semi-definite potential \( V(\phi) \):

\[
\mathcal{L} = \frac{1}{2} G_{ab}(\phi) \partial_\mu \phi^a \partial^\mu \phi^b - V(\phi),
\]

(B.1)

where \( G_{ab}(\phi) \) is also a positive-semi-definite function of \( \phi \). For static condition, \( \phi(x) \), the total energy functional is

\[
E(\phi) = \int d^d x \left( \frac{1}{2} G(\phi) \nabla \phi \cdot \nabla \phi + V(\phi) \right)
\]

\[
\equiv E_k + E_p,
\]

(B.2)

where subscripts denote the kinetic and potential parts of the energy. Since \( G(\phi) \) and \( V(\phi) \) are both functions of \( \phi \) only, the kinetic part has at most quadratic power in spatial derivative while the potential term has no at all.

Now consider a spatial rescaling of \( x \), \( i.e., \) a mapping \( x \to \mu x \) with \( \mu \geq 0 \). Scalar field \( \phi \) simply transforms as

\[
\phi(x) \to \phi(\mu x),
\]

(B.3)

\footnote{Since in general \( G_{ab}(\phi) \) is not constant, this theorem applies also to non-linear sigma model.}
but the field gradient becomes
\[ \nabla \phi(x) \to \mu \nabla \phi(\mu x). \]  
(B.4)

The rescaled energy then
\[ E^{(\mu)}(\phi) = \int \left( \mu^{2-d} G(\phi) \nabla \phi \cdot \nabla \phi + V(\phi) \right) \mu^{-d} d^d x \]
\[ = \mu^{2-d} E_k^{(\mu)} + \mu^{-d} E_p^{(\mu)}, \]  
(B.5)
where we have used the fact that \( d^d x \to \mu^{-d} d^d x \) under the spatial rescaling.

Derrick’s theorem formally says

“Suppose that for an arbitrary finite-energy field configuration \( \phi(x) \), which is not in the vacuum, the function \( E^{(\mu)} \) has no stationary point. Then the theory has no static solutions of the field equation having finite energy other than the vacuum.”

The statement that energy is minimum under the spatial rescaling means that the following stationary condition holds:
\[ \frac{dE}{d\mu} \bigg|_{\mu=1} = 0, \]  
(B.6)
and, moreover, the stability is guaranteed if that stationary point is a stable minima, \( i.e., \)
\[ \frac{d^2E}{d\mu^2} \bigg|_{\mu=1} > 0. \]  
(B.7)

We can easily see that unless in \( d = 1 \), under no circumstances the system described by theory (B.1) satisfies conditions (B.6)-(B.7), \( i.e., \) that no finite-energy defects are stable in any pure-scalar-field theory\(^2\). The case \( n = 1 \) is what we already encountered before, the \textit{kink} solutions. The stability condition above shows that in order to have a definite size, the kink should satisfy the energy condition \( E_k = E_p \).

\(^2\)The case for \( n = 2 \) is quite subtle. Since the rescaled energy is independent of \( \mu \), \( E^{(\mu)} = E_k \), the theory is conformally-invariant. There exists a metastable-finite-energy solution \([159]\) in this theory, which should properly be classified as \textit{lumps} rather than solitons. This is an example of sigma-model in \( 2 + 1 \)-dimensions.
This unfortunate condition (of defect-non-existence) can be cured if we enrich the theory, \textit{i.e.}, we allow more fields to play role. In fact, Derrick himself in his paper \cite{14} gives some suggestions to circumvent the non-existence problems:

- by allowing the system to be time-dependent, \textit{i.e.}, $\phi(t, x)$.
- by including more fields other than just scalar.
- and by considering non-canonical kinetic term(s), \textit{i.e.}, making $G(\phi, \partial_\mu \phi)$.

The first suggestion is discussed by Coleman \cite{5} and Lee and Pang \cite{4} in the context of non-topological solitons. They found that by having internal “rotations”, stable solitons can exist.

The second suggestion is realized in local defects. The existence of gauge fields\footnote{Non-abelian, in general.} $A^a_\mu$ helps evading Derrick’s theorem. This is because the gauge fields transform as

$$A^a_\mu(x) \to \mu A^a_\mu(\mu x)$$

under spatial rescaling. This behavior can be traced back to the fact that gauge fields are 1-form potentials, and its existence is to covariantize the ordinary derivative so that the theory is invariant under gauge transformations,

$$\partial_\mu \phi^a \to D_\mu \phi^a \equiv \partial_\mu \phi^a + e e^{abc} A^b_\mu \phi^c.$$  \hfill (B.9)

From dimensional analysis we can easily see that $A^a$ should behave the same way $\nabla$ does under the rescaling. Further, the field strength $F^a_{\mu\nu}$ behaves

$$F^a_{\alpha\beta}(x) \to \mu^2 F^a_{\alpha\beta}(\mu x).$$ \hfill (B.10)
Inserting the gauge-field term into the energy functional yields

\[
E(\phi) = \int d^d x \left( \frac{1}{4} F^2 + \frac{1}{2} G(\phi) |\nabla \phi|^2 + U(\phi) \right)
\equiv E_g + E_k + E_p, \quad (B.11)
\]

where the subscript \( g \) denotes the contribution of energy coming from the
gauge field. Now the key point in having the gauge fields lies on the fact that
its energy term goes like fourth-power of the spatial derivative, \([F]^2 \sim [\nabla]^4\).
This gives the rescaled energy

\[
E^{(\mu)} = \mu^{4-d} E_g^{(\mu)} + \mu^{2-d} E_k^{(\mu)} + \mu^{-d} E_p^{(\mu)}. \quad (B.12)
\]

It is now trivial to see that conditions (B.6)-(B.7) can be satisfied for \( d = 2 \)
and \( d = 3 \); they correspond to local strings and magnetic monopoles, respec-
tively. Gauged strings (or vortices) can be abelian (Abelian-Higgs theory) or
non-abelian. Magnetic monopoles is necessarily non-abelian, Yang-Mills-Higgs
theory.

The last suggestion by Derrick can be realized if we consider higher powers
of derivatives of \( \phi \) in (B.1). Indeed, Skyrme model is one famous example of
this. Skyrme added a fourth-power of derivative of \( \phi \) in the Lagrangian and
obtained a stable-finite-energy solitons, skyrmions.

Finally, it is amusing to consider that the Yang-mills-Higgs Lagrangian in
\( d = 4 \) does not give any finite-energy solutions, but the pure Yang-Mills does.
If we consider the system described by

\[
\mathcal{L} = -\frac{1}{4} \int d^4 x F_{\mu\nu}^a F^{a\mu\nu}, \quad (B.13)
\]

then the rescaled energy is

\[
E^{(\mu)} = \mu^{4-d} E_g. \quad (B.14)
\]
In 4-dimensional Euclidean space it is conformally-invariant and finite-energy solutions are possible. Indeed such solutions for winding number unity have been successfully constructed by Belavin \textit{et al} \cite{160} and are dubbed \textit{instantons}.\footnote{They originally named it \textit{pseudoparticles}. The term \textit{instantons} was coined by ’t Hooft to describe a particle-like solutions (in a would-be five-dimensional space-time) which are localized in \textit{instantaneous} Euclidean-time and space.} The constructions for multi-instantons solutions are a non-trivial task and attract the physicists and mathematicians alike due to its mathematical beauty \cite{161}.
Appendix C

Bogomolny-Prasad-Sommerfield Condition

To obtain the full profile solutions of $p(r)$ and $w(r)$ functions in Yang-Mills-Higgs theory we have to solve the Euler-Lagrange equations of Lagrangian (1.13)

$$p'' + \frac{2p'}{r} - \frac{2w^2 p}{r^2} - \lambda \eta^2 p(p^2 - 1) = 0,$$
$$w'' + \frac{w(1 - w^2)}{r^2} - e^2 \eta^2 p^2 w = 0. \quad (C.1)$$

In general no analytic solutions have been found though numerical solutions are straightforward to solve using shooting method (for example, see [12]).

Remarkably, there exist analytic solutions for $\lambda = 0$. Shortly after the discovery of magnetic monopoles by 't Hooft and Polyakov, Prasad and Sommerfield [162] tried to solve the equations of motion in the limit of vanishing Higgs potential. Initially they were fitting some analytic functions to approximate the solutions by trial and error. Apparently what they obtained was in
fact exact solutions! They found

\[ p(r) = \coth(m_v r) - \frac{1}{m_v r}, \]

\[ w(r) = \frac{m_v r}{\sinh(m_v r)}. \]  

(C.2)  

(C.3)

This serendipitous discovery receives a deeper understanding from Bogomolny’s analysis [10]. Based on energy consideration he independently derived the same exact solutions and interpreted it as global minimum configurations. In the case \( \lambda = 0 \) the total energy (or mass of the monopole) reads

\[ E = \int d^3 x \frac{1}{2} (B^a \cdot B^a + D\phi^a \cdot D\phi^a). \]  

(C.4)

By the same smart method of completing the square, we can re-arrange the right-hand side as

\[ E = \int d^3 x \frac{1}{2} (B^a - D\phi^a)^2 + \int d^3 x B^a \cdot D\phi^a, \]

\[ = \int d^3 x \frac{1}{2} (B^a - D\phi^a)^2 + \oint_{\Sigma} dS \cdot B^a \phi^a, \]  

(C.5)

where on the second line we have employed Bianchi identity, \( D_\mu \tilde{F}^{\mu \nu} = 0 \), and Stokes theorem, \( \int d^3 x D \cdot A = \oint_{\Sigma} dS \cdot A \), to derive the second term on the right-hand side. Now, since the first term is positive-definite, the energy is bounded from below

\[ E \geq \oint_{\Sigma} dS \cdot B^a \phi^a, \]

\[ \geq \frac{4\pi}{e^2} m_v, \]  

(C.6)

the Bogomolny bound. This bound is saturated if and only if

\[ B^a = D\phi^a. \]  

(C.7)
As in the case with domain walls the Bogomolny equation is first-order, and he showed that the solutions are exactly the same as eq. (C.2). From this energy-bound analysis we can see that the Bogomolny-Prasad-Sommerfield (BPS) solutions are the global minimum of the energy; they are globally stable. Moreover, the lower bound is proportional to the monopole charge[^1].

As \( r \to \infty \), the scalar field function \( p(r) \) goes like

\[
p(r) \to 1 - \frac{1}{m_v r} + \mathcal{O}(e^{-m_v r}),
\]

(C.8)

with Coulombian tail dominates at large distance. This behavior is easy to understand if we recall that with vanishing \( \lambda \) the Higgs field becomes massless, \( m_H = 0 \). It is this Goldstone boson nature that gives rise to the dominant-power-law behavior far away from the core. This brings an interesting implications for multi-monopoles configurations. Static multi-monopoles are allowed to exist in BPS condition because the repulsive force of Yang-Mills field is compensated by the attractive (Coulomb) force of the Higgs field.

[^1]: In this case, \( n = 1 \).
Appendix D

Graviton Perturbations

Though unsuccessful, it is instructive to review the derivation of the equation of fluctuations that might give rise to the metastable graviton trapping on the brane albeit with infinite-volume extra dimensions, i.e., the Dvali-Gabadadze-Porrati (DGP) mechanism \[63\]. Since the method presented here is general without appealing to a particular bulk field, the hope is that this derivation can be used for the future work that accommodates more ingredients in the theory (e.g., scalar fields with non-minimal coupling to gravity \[115\] \[116\]).

To do this, let us consider the metric fluctuation

\[
\eta_{\mu\nu} \rightarrow \eta_{\mu\nu} + h_{\mu\nu}, \tag{D.1}
\]

\[h_{\mu\nu} = h_{\mu\nu}(x^\alpha, r, \theta_i),\] and we choose to work on TTF (transverse-tracefree) gauge, i.e. \(h_{\mu\nu}\) satisfies

\[
\begin{align*}
h_\alpha &= 0 \\
\nabla_\mu h^{\mu\nu} &= 0. \tag{D.2}
\end{align*}
\]
Since \( g_{\mu\nu} = B^2 \eta_{\mu\nu} \), we have

\[
g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu},
\]

(D.3)

with

\[
\delta g_{\mu\nu} = B^2 h_{\mu\nu},
\]

\[
\delta g^{\mu\nu} = -2 B^2 \eta^{\mu\alpha} \eta^{\nu\beta} h_{\alpha\beta}.
\]

(D.4)

The perturbed Christoffel Symbols, defined as

\[
\delta \Gamma^A_{BC} \equiv \frac{1}{2} \delta g^{AD} (g_{DB,C} + g_{DC,B} - g_{BC,D}) + \frac{1}{2} g^{AD} [(\delta g_{DB})_C + (\delta g_{DC})_B - (\delta g_{BC})_D],
\]

(D.5)

are

\[
\delta \Gamma^\alpha_{\mu\nu} = \frac{1}{2} \eta^{\alpha\beta} (h_{\beta\mu,\nu} + h_{\beta\nu,\mu} - h_{\mu\nu,\beta}),
\]

\[
\delta \Gamma^r_{\mu\nu} = \frac{B}{H^2} h_{\mu\nu} + \frac{B^2}{2H^2} h'_{\mu\nu},
\]

\[
\delta \Gamma^\mu_{r\nu} = \frac{1}{2} h'_{\nu},
\]

\[
\delta \Gamma^\theta_i_{\mu\nu} = \frac{B^2}{2Y_i H^2 r^2} \partial_\theta h_{\mu\nu},
\]

\[
\delta \Gamma^\mu_{\theta_i\nu} = \frac{1}{2} \partial_\theta h'_{\nu}.
\]

(D.6)

The perturbed Ricci tensor is

\[
\delta R_{\mu\nu} = (\delta \Gamma^A_{\mu\nu})_A - (\delta \Gamma^A_{\muA})_{\nu} + (\delta \Gamma^A_{\mu\nu})_{AB} \Gamma^B_{\muA} + (\delta \Gamma^A_{\mu\nu})_{\mu\nu} (\delta \Gamma^B_{AB}) - (\delta \Gamma^B_{\muA}) \Gamma^A_{B\nu} - \Gamma^A_{\mu\nu}(\delta \Gamma^A_{B\nu}),
\]

(D.7)

which, by virtue of (D.6) and (2.50), yields

\[
\frac{\delta R_{\mu\nu}}{B^2/2H^2} = h''_{\mu\nu} + \left( (p + 1) \frac{B'}{B} + (D - 2) \frac{H'}{H} + \frac{(D - 1)}{r} \right) h'_{\mu\nu}
\]

\[
+ 2 \left( \frac{B''}{B} + (d - 2) \frac{B'H'}{BH} + (D - 1) \frac{B'}{Br} + \frac{B'^2}{B^2} \right).
\]

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the last expression is made possible by the Einstein’s equation \[ R_0^0 = 0 \]. One then arrives at
\[
\begin{aligned}
&h''_{\mu\nu} + \left[ (p + 1) \frac{B'}{B} + (D - 2) \frac{H'}{H} + \frac{(D - 1)}{r} \right] h'_{\mu\nu} \\
&+ \left[ \frac{1}{r^2} L_{D-1} - \frac{H^2}{B^2} \Box \right] h_{\mu\nu},
\end{aligned}
\]
(D.8)

where \( \Box \equiv \eta^{\mu\nu} \partial_\mu \partial_\nu \) is the four-dimensional d’Alembertian, and \( L_{d-1} \) stands for the hyperspherical Laplacian,
\[
L_{d-1} \equiv \sum_{i=1}^{d-1} \frac{1}{\Upsilon_i} \left( \partial^2_{\theta_i} + \frac{d - i - 1}{\tan \theta_i} \partial_{\theta_i} \right).
\]
(D.10)

By performing Fourier transformation and (hyper)-spherical harmonics decomposition on the metric fluctuation, we could extract out [115, 98, 116]
\[
\Box \rightarrow -M^2,
\]
\[
L_{d-1} \rightarrow \ell(\ell + d - 2).
\]
(D.11)

We can also, for convenience, conformally transform the radial coordinate by defining \( z \) such that,
\[
z \equiv \int_0^r d\rho \frac{H(\rho)}{B(\rho)};
\]
also known as the tortoise coordinate. This change of coordinate transforms

\[ ^1 \text{Notice that we consider only the perturbed vacuum Einstein’s equations. This is because we are interested in the effect of the bulk matter on the metric fluctuations } h_{\mu\nu}, \text{ not in the back-reaction of this fluctuation on the matter.} \]
the graviton equation into

\[ \frac{d^2 h_{\mu\nu}}{dz^2} + \left[ \frac{p B'}{H} + (D-1) \left( \frac{H'B}{H^2} + \frac{B}{Hr} \right) \right] \frac{dh_{\mu\nu}}{dz} + \left( \frac{B^2 H^2 r^2 (\ell + \ell + 2) + M^2}{H^2 r^2} \right) h_{\mu\nu} = 0. \]  
(D.13)

Furthermore, by defining

\[ \xi_{\mu\nu} \equiv B^2 (Hr)^{(D-1)/2} h_{\mu\nu}, \]  
(D.14)

we are able to rewrite (D.9) into a Schrödinger-like form

\[ -\frac{d^2 \xi_{\mu\nu}}{dz^2} + U(r) \xi_{\mu\nu} = M^2 \xi_{\mu\nu}, \]  
(D.15)

where

\[ U(r) \equiv V(r)^2 + V'(r) + \frac{B^2}{H^2 r^2} \ell (\ell + d + 2), \]  
(D.16)

with

\[ V(r) \equiv \frac{p B'}{2H} + \frac{(D-1)}{2} \left( \frac{H'B}{H^2} + \frac{B}{Hr} \right), \]  
(D.17)

and \( r \) is to be understood as a function of \( z \), \( r = r(z) \).

Thus, by studying the depth and the barrier thickness of Schrödinger potential \( V(r) \) one can determine the existence of a metastable graviton. Such an existence can realize 4d Newtonian gravity on intermediate length scale on the brane, while at both short- and long-range gravity becomes higher-dimensional [115, 116].
Appendix E

Dimensional Reduction of (N + n + 1)-D Action

Consider a general Action

$$S = \int d^{N+n+1}x \sqrt{-\hat{G}} \left[ \frac{1}{16\pi G_{N+n+1}} R^{(N+n+1)}(\hat{G}_{AB}) + \mathcal{L}_m \right], \quad (E.1)$$

with $\mathcal{L}_m$ any bulk fields Lagrangian, $N$ the number of large spatial dimensions, and $n$ the number of compact extra dimensions\footnote{We follow the analysis of \cite{163, 164, 128}.}. The metric $\hat{G}_{AB}$ is given by

$$ds^2 = \hat{G}_{AB} dx^A dx^B,$$

$$= g_{\mu\nu}(x) dx^\mu dx^\nu + b^2(x) \gamma_{ij} dy^i dy^j, \quad (E.2)$$

with $\mu, \nu = 0, 1, 2, ..., N$ run through the large dimensions, $i, j = N+1, ... N+n$ denote the compact extra dimensions, and $b(x)$ the \textit{radion} that characterizes the size of extra dimensions.

\footnote{At the end we will take $N = 3$ and $n = 2$. For the case relevant in this dissertation $\mathcal{L}_m$ is the Maxwell Lagrangian}
The \((N + n + 1)\) scalar curvature can be decomposed into
\[
R^{(N+n+1)} = \hat{G}^{AB} R^{(N+n+1)}_{AB},
\]
\[
= g^{\mu\nu} R^{(N+1)}_{\mu\nu}(g_{\alpha\beta}, b, \gamma_{lm}) + b^{-2} \gamma^{ij} R^{(n)}_{ij}(g_{\alpha\beta}, b, \gamma_{lm}).
\] (E.3)

By evaluating the Christoffel symbols of metric (E.2) we can express the \((N + 1)\) and \(n\) dimensionals Ricci tensors as, respectively,
\[
R^{(N+1)}_{\mu\nu}(g_{\alpha\beta}, b, \gamma_{lm}) = R^{(N+1)}_{\mu\nu}(g_{\alpha\beta}) - \frac{n \nabla_{\mu} \nabla_{\nu} b}{b};
\]
\[
R^{(n)}_{ij}(g_{\alpha\beta}, b, \gamma_{lm}) = R^{(n)}_{ij}(\gamma_{lm}) - \gamma_{ij} b \nabla_{\mu} \nabla^{\mu} b - (n - 1) \gamma_{ij} (\nabla_{\mu} b)(\nabla^{\mu} b).
\] (E.4)

The volume of extra dimensions is
\[
\mathcal{V} = \int d^n y \sqrt{\gamma},\] (E.5)

using which we can define the higher-dimensional gravitational constant as
\[
G_{(N+n+1)} = \mathcal{V} G_{N+1}.
\] (E.6)

Assuming that the extra dimensions are maximally-symmetric, we can write its Ricci scalar as
\[
R^{(n)}(\gamma_{ij}) = \frac{n(n-1)}{R_0^2},
\] (E.7)

with \(R_0\) the radius of extra dimensions. Combining them, the Action (E.1) can be re-written as
\[
S = \int d^{N+1} x \sqrt{-g} \left[ \frac{1}{16 \pi G_{N+1}} \left( b^n R^{(N+1)} + b^{n-2} \frac{n(n-1)}{R_0^2} ight) 
- 2 n b^{n-1} g^{\mu\nu} (\nabla_{\mu} b)(\nabla_{\nu} b) - n(n-1) b^{n-2} g^{\mu\nu} (\nabla_{\mu} b)(\nabla_{\nu} b) 
+ b^n \mathcal{V} \mathcal{L}_m \right].
\] (E.8)

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The Action (E.8) is a dimensionally-reduced form of (E.1). However, the Ricci scalar is directly coupled to the radion; this Action takes the form of Brans-Dicke-type (also known as Jordan frame or string frame) rather than Einstein-Hilbert-type. To bring this Action into the Einstein frame, we redefine

\[ b(x) \equiv e^{\beta(x)}, \quad \text{(E.9)} \]

and Weyl-rescale (i.e., conformally transform) the metric

\[ g_{\mu\nu} \rightarrow e^{-\frac{2n\beta(x)}{N-1}} g_{\mu\nu}. \quad \text{(E.10)} \]

Under these re-definitions, the Ricci scalar transforms (after a lengthy algebra) as

\[
\begin{align*}
R^{(N+1)}_{\mu\nu}(g_{\alpha\beta}) & \rightarrow e^{\frac{2n\beta}{N-1}} R^{(N+1)} + e^{\frac{2n\beta}{N-1}} n g^{\mu\nu} \nabla_\mu (\nabla_\nu \beta) \\
& + e^{\frac{2n\beta}{N-1}} n \left[ (N + 1) \nabla_\alpha (\nabla^\alpha \beta) + n g^{\mu\nu} (\nabla_\mu \beta)(\nabla_\nu \beta) \\
& - n(N + 1) (\nabla_\alpha \beta)(\nabla^\alpha \beta) \right].
\end{align*}
\quad \text{(E.11)}
\]

Now the radion decouples from the curvature scalar and the Action takes the conventional Einstein’s gravity form, given by

\[
S = \int d^{N+1}x \sqrt{-g} \left[ \frac{1}{16\pi G_{N+1}} \left( R^{(N+1)} - \frac{n(N + n - 1)}{N - 1} g^{\mu\nu} (\nabla_\mu \beta)(\nabla_\nu \beta) \\
+ \frac{n(n - 1)}{R_0^2} e^{-\frac{2n\beta}{N-1}(N+n-1)} \right) + \mathcal{V} e^{-\frac{2n\beta}{N-1}} L_m \right].
\quad \text{(E.12)}
\]

If we define

\[
M_P \equiv \kappa^{-1} \equiv \sqrt{\frac{1}{8\pi G_{N+1}}}, \quad \psi \equiv \sqrt{\frac{n(N + n - 1)}{N - 1}} M_P \beta,
\quad \text{(E.13)}
\]
we can bring the Action (E.11) into the canonical expression

\[ S = \int d^{N+1}x \sqrt{-g} \left[ \frac{1}{2\kappa^2} R - \frac{1}{2} g^{\mu\nu} (\nabla_\mu \psi)(\nabla_\nu \psi) \right. \\
+ \frac{1}{2} \frac{n(n-1)}{R_0^2 \kappa^2} e^{-2\sqrt{\frac{R_0}{n(N-1)}} \frac{\psi}{M_P}} + \mathcal{V} e^{-2\sqrt{\frac{R_0}{(N-1)(N+n-1)}} \frac{\psi}{M_P}} \mathcal{L}_m \right]. \]  

(E.14)

For \( N = 3 \) and \( n = 2 \), the radion and the metric reduces to

\[ b^2 = e^{\psi/M_P}, \]

\[ g_{\mu\nu} \to e^{-\psi/M_P} g_{\mu\nu}. \]  

(E.15)

The Maxwell Lagrangian for compactification solutions is, upon the rescaling,

\[ \mathcal{L}_m = \frac{-1}{2e^2 R_0} e^{-2\psi/M_P} - \Lambda. \]  

(E.16)

This makes the Action (E.14) becomes

\[ S = \int d^{N+1}x \sqrt{-g} \left[ \frac{1}{2\kappa^2} R - \frac{1}{2} g^{\mu\nu} (\nabla_\mu \psi)(\nabla_\nu \psi) - V(\psi) \right], \]  

(E.17)

with \( V(\psi) \) the 4d effective potential of the radion, given by

\[ V(\psi) = 4\pi \left( \frac{1}{2e^2 R_0^2} e^{-3\psi/M_P} - \frac{M_P^2}{4\pi R_0^2} e^{-2\psi/M_P} + R_0^2 \Lambda e^{-\psi/M_P} \right). \]  

(E.18)
Appendix F

$n = 1$ Compactification Solutions

For compactifications of unit flux number, Eqs. (3.17), (3.18), and (3.19) reduce to

\[
3H^2 + \frac{1}{C^2} = \kappa^2 \left( \frac{p^2 w^2}{C^2} + \frac{(1 - w^2)^2}{2e^2C^4} + \frac{\lambda}{4}(p^2 - \eta^2)^2 + \Lambda \right),
\]

\[
6H^2 = \kappa^2 \left( - \frac{(1 - w)^2}{2e^2C^2} + \frac{\lambda}{4}(p^2 - \eta^2)^2 + \Lambda \right),
\]

\[
0 = \frac{2pw^2}{C^2} + \lambda \ p \ (p^2 - \eta^2),
\]

\[
0 = \frac{w(1 - w^2)}{C^2} - e^2 p^2 w. \quad \text{(F.1)}
\]

This is a system of four algebraic quadratic equations for four unknown functions: $H, C, p,$ and $w$. In general there are $4^2 = 16$ solutions for each of them, however not all of them are physical (e.g., $p < 0$ and/or $w < 0$). The physical ones are discussed below. They can be classified into stable and unstable solutions. Obviously in the body of this dissertation we only work with the stable solutions. But for completeness, we also construct unstable solutions. However, for simplicity we rescale $\eta p \rightarrow p$. With this notation, the scalar field is in the vacuum manifold when $p = \eta$. 

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1For simplicity we rescale $\eta p \rightarrow p$. With this notation, the scalar field is in the vacuum manifold when $p = \eta$. 

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configurations which are interesting in their own right\textsuperscript{2}.

\section*{F.1 Stable Compactification Solutions}

Stable solutions exist when the fields relax to their vacuum values, \( p = \eta \), \( w = 0 \), yielding

\[
H^2 = \frac{2\kappa^2\Lambda}{9} - \frac{2e^2}{27\kappa^2} \left( 1 + \sqrt{1 - \frac{3\kappa^4\Lambda}{2e^2}} \right),
\]

\[
C^2 = \frac{1}{\kappa^2\Lambda} \left( 1 - \sqrt{1 - \frac{3\kappa^4\Lambda}{2e^2}} \right). \quad (F.2)
\]

One can see that this is a stable configuration by looking at the 4d effective action about this solution. These are the most interesting solutions for our purpose, although one can find several other solutions which are unstable.

\section*{F.2 Unstable Compactification Solutions}

\subsection*{F.2.1 Compactifications with \( p = \eta \) and \( w = 0 \)}

Here the matter fields have relaxed to their respective vacua, but \( C \) is sitting at an unstable equilibrium for the size of the compactification manifold. The solutions take the form

\[
H^2 = \frac{2\kappa^2\Lambda}{9} - \frac{2e^2}{27\kappa^2} \left( 1 - \sqrt{1 - \frac{3\kappa^4\Lambda}{2e^2}} \right),
\]

\[
C^2 = \frac{1}{\kappa^2\Lambda} \left( 1 + \sqrt{1 - \frac{3\kappa^4\Lambda}{2e^2}} \right). \quad (F.3)
\]

\textsuperscript{2}The first type of unstable solutions presented here may also occur for \( n > 1 \), although none of the subsequent examples generalize in this way.
F.2.2 Compactification with \( p = 0 \) and \( w = 1 \)

These configurations are clearly unstable since \( p = 0 \) and \( w = 1 \) implies that the Higgs and gauge fields are turned off.

\[
\begin{align*}
H^2 & = \frac{\kappa^2}{24}(\lambda \eta^4 + 4\Lambda), \\
C^2 & = \frac{8}{\kappa^2(\lambda \eta^4 + 4\Lambda)}.
\end{align*}
\]  

(F.4)

This is a pure 6d gravity theory, and without any flux stabilizing the extra dimensions the radion \( C \) will grow with time and soon this configuration decompactify to 6d de Sitter space.

F.2.3 Compactification with \( p = \eta \) and \( w = 1 \)

This “compactification” corresponds to the 6d Einstein-Higgs theory. In fact, there is no compactification solution at all. It is easy to see that unless

\[
\begin{align*}
H^2 & = \sqrt{\frac{\Lambda_0}{6}}, \\
C^2 & \to \infty,
\end{align*}
\]  

(F.5)

the system of equations (F.1) are not self-consistent with \( p = \eta \) and \( w = 1 \).

F.2.4 Compactification with \( p = 0 \) and \( w = 0 \)

The Higgs field is turned off and we are left with gauge field only for the flux source. This is a case of 6d Einstein-Yang-Mills flux compactification. It has been shown by Randjbar-Daemi et al [165] that this theory exhibits tachyonic excitations and is thus unstable.

To write down the solutions, we define

\[
\beta \equiv \sqrt{e^2(8e^2 - 3\kappa^4(\eta^4\lambda + 4\Lambda_0))},
\]  

(F.6)
such that $H$ and $C$ are given by

\[ H^2_{\pm} = \frac{-4e^2 + 3\kappa^4(\eta^4\lambda + 4\Lambda_0) \pm \sqrt{2}\beta}{54\kappa^2}, \]

\[ C^2_+ = \frac{6\kappa^2}{-4e^2 + \sqrt{2}\beta}, \]

\[ C^2_- = \frac{-4e^2 + \sqrt{2}\beta}{e^2\kappa^2(\eta^4\lambda + 4\Lambda_0)}. \]  

(F.7)

F.2.5 Non-zero constant $p$-$w$ Solutions

This is a solution where both $p$ and $w$ are non-zero constants different from their vacuum values. We have checked numerically that this type of solution is unstable to decompactification, as was indicated in [166].

To simplify the notation, we define the quantity

\[ \alpha = \sqrt{(8e^2 + 4\lambda(\eta_2k^2 - 1))^2 - 24\kappa^2(e^2(\lambda\eta^4 + 4\Lambda) - 2\lambda\Lambda)}, \]  

(F.8)

allowing the solutions to be written

\[ p = \sqrt{\frac{-4e^2 + 2\lambda + \lambda\eta^2\kappa^2 \pm \frac{1}{2}\alpha}{(3\lambda - 6e^2)}}, \]

\[ w = \sqrt{\frac{\lambda(43^2\eta^2 + 2\kappa^2\lambda\Lambda + e^2(\lambda\eta^4\kappa^2 - 2\lambda\eta^2 - 4\kappa^2\Lambda)) \pm \frac{1}{2}e^2\eta^2\alpha}{\kappa^2(2e^2 - \lambda)\left(e^2(\lambda\eta^4 + 4\Lambda) - 2\lambda\Lambda\right)}}, \]

\[ H^2 = \frac{\kappa^2\left(e^2(\lambda\eta^4 + 4\Lambda) - 2\lambda\Lambda\right)\left(2e^2 - \lambda + \eta^2\kappa^2\lambda \pm \frac{1}{2}\alpha\right)}{9(2e^2 - \lambda)\left(4e^2 - 2\lambda + 2\eta^2\kappa^2\lambda \pm \frac{1}{2}\alpha\right)}, \]  

(F.9)

\[ C^2 = \frac{6\kappa^2}{4e^2 - 2\lambda + 2\lambda\eta^2\kappa^2 \mp \frac{1}{2}\alpha}. \]